J. London Math. Soc. (2) 94 (2016) 933–969
doi:10.1112/jlms/jdw065

The augmented marking complex of a surface

Matthew Gentry Durham

Abstract

We build an augmentation of the Masur–Minsky marking complex by Groves-Manning combinatorial horoballs to obtain a graph we call the *augmented marking complex*, $\mathcal{AM}(S)$. Adapting work of Masur–Minsky, we show that this augmented marking complex is quasiisometric to Teichmüller space with the Teichmüller metric. A similar construction was independently discovered by Eskin–Masur–Rafi. We also completely integrate the Masur–Minsky hierarchy machinery to $\mathcal{AM}(S)$ to build flexible families of uniform quasigeodesics in Teichmüller space. As an application, we give a new proof of Rafi's distance formula for $\mathcal{T}(S)$ with the Teichmüller metric. We have included an appendix, in which we prove a number of facts about hierarchies that we hope will be of independent interest.

1. *Introduction*

The study of various combinatorial complexes built from simple closed curves on surfaces has greatly advanced the state of knowledge of the geometry of Teichmüller space, $\mathcal{T}(S)$, the mapping class group, $\mathcal{MCG}(S)$, and hyperbolic 3-manifolds. In [**7**], Brock showed that $\mathcal{T}(S)$ with the Weil–Petersson metric is quasiisometric to the graph of pants decompositions on S, $\mathcal{P}(S)$, an insight which he used to prove that the Weil–Petersson distance between two points in $\mathcal{T}(S)$ is coarsely the volume of the convex core of the quasi-Fuchsian hyperbolic 3-manifold they simultaneously uniformize. Beginning with their proof of hyperbolicity of the curve complex, $\mathcal{C}(S)$, in [**19**], the hierarchy machinery Masur–Minsky developed in [**20**] was essential in the proof of the Ending Lamination Theorem [**8**, **23**] for hyperbolic 3-manifolds. Moreover, in [**20**], Masur–Minsky built the marking complex, $\mathcal{M}(S)$, and prove it is quasiisometric to $\mathcal{MCG}(S)$ in any word metric, an analogy essential to the proofs of the rank conjecture [**5**] and quasiisometric rigidity [**4**] theorems for the mapping class group.

The main goal of this paper is to build a combinatorial complex, the *augmented marking complex*, which is quasiisometric to $\mathcal{T}(S)$ in the Teichmüller metric.

Theorem 1.1. *The augmented marking complex, $\mathcal{AM}(S)$, is $\mathcal{MCG}(S)$-equivariantly quasiisometric to $\mathcal{T}(S)$ in the Teichmüller metric.*

A large part of this paper is spent adapting the Masur–Minsky hierarchy machinery for $\mathcal{M}(S)$ and $\mathcal{P}(S)$ to $\mathcal{AM}(S)$. We use these augmented hierarchies for $\mathcal{AM}(S)$ to build families of uniform quasigeodesics called *augmented hierarchy paths*, and derive a version of Rafi's distance formula for the Teichmüller metric (Theorem 2.10), thereby completing the unification of the coarse geometries of $\mathcal{MCG}(S)$ and $\mathcal{T}(S)$ in the Weil–Petersson, and Teichmüller metrics by a common framework developed in [**7**, **19**, **20**, **24**, **25**]. In a recent paper, Eskin–Masur–Rafi [**12**] used $\mathcal{AM}(S)$ and augmented hierarchy paths, which they independently discovered,

Received 28 December 2013; revised 10 November 2015; published online 21 October 2016.

2010 *Mathematics Subject Classification* 30F60 (primary), 20F65 (secondary).

The author was partially supported by the National Science Foundation under Grant Number NSF 1045119.

to prove the Brock-Farb Geometric Rank Conjecture for $\mathcal{T}(S)$ with the Teichmüller metric (see [**9**]). Bowditch [**6**], Behrstock–Hagen–Sisto [**3**], and the author [**11**] have also used $\mathcal{AM}(S)$ to give different, independent proofs of the rank conjecture.

Our construction follows upon the work of Masur and Minsky on the curve and marking complexes [**19**, **20**], and Rafi's applications of their machinery to Teichmüller geometry [**24**, **25**], though we emphasize that our work is independent of Rafi's. We now briefly discuss the context of these results.

The Teichmüller space of a surface S, denoted by $\mathcal{T}(S)$, is the space of hyperbolic metrics on S up to isotopy. The geometry of the thin part of $\mathcal{T}(S)$, those metrics for which the hyperbolic lengths of some curves on the surface are small, is fundamentally different from its complement, the thick part. One can see this in the completion of $\mathcal{T}(S)$ in the Weil–Petersson metric, where curves are pinched to nodes, and the geometry of the boundary strata is that of a product of the Teichmüller spaces of the complements of the pinched curves. While this stark phenomenon does not exactly hold in the Teichmüller metric, Minsky proved in [**22**] that the Teichmüller metric on the thin part of $\mathcal{T}(S)$ is quasiisometric to the product of the Teichmüller spaces of the complements of the short curves and a product of horodisks, one for each short curve (see Theorem 2.8) with the sup metric; that is, the thin parts of $\mathcal{T}(S)$ coarsely have a product structure.

In [**19**], Masur and Minsky proved that Harvey's complex of simple closed curves [**15**] on S, denoted by $\mathcal{C}(S)$, is δ-hyperbolic, and that the electrification of the thin parts of $\mathcal{T}(S)$ is quasi-isometric to $\mathcal{C}(S)$, and thus hyperbolic. While this provides for a substantial amount of control over the large-scale geometry of $\mathcal{C}(S)$ and the thick part of $\mathcal{T}(S)$, $\mathcal{C}(S)$ is locally infinite, whereas $\mathcal{T}(S)$ is proper with the Teichmüller metric, and thus hyperbolicity does little *a priori* to inform upon the local geometry of either. In [**20**], they introduced the machinery of *hierarchies* of tight geodesics, which record the combinatorial information sufficient to gain a great deal of control over the local geometry of $\mathcal{C}(S)$, proving it shares some properties with locally finite complexes. These hierarchies also contain the information sufficient to build quasigeodesics in the associated marking complex, $\mathcal{M}(S)$, called *hierarchy paths*. They proved that the progress along a hierarchy path coarsely occurs in subsurfaces to which the end markings have heavily overlapping projections. Using the hierarchy machinery, they proved that $\mathcal{M}(S)$ is $\mathcal{MCG}(S)$-equivariantly quasiisometric to $\mathcal{MCG}(S)$ with any word metric, and obtained a coarse distance formula for $\mathcal{MCG}(S)$ (Theorem 2.7).

The connection between the work of Masur–Minsky and the Teichmüller metric was largely developed by Rafi; see [**26**] for a summary of the current state of this project. A Teichmüller geodesic is a path through a space of metrics on S, and one may ask when a given curve $\alpha \in \mathcal{C}(S)$ is shorter than some fixed constant. In [**24**], Rafi proved that the hyperbolic length of a curve along a Teichmüller geodesic, $\mathcal{G}$, is shorter than the constant from Minsky's Product Regions theorem (Theorem 2.8) at some point along $\mathcal{G}$, if the vertical and horizontal foliations which determine $\mathcal{G}$ heavily overlap on a subsurface of which that curve is a boundary component. In its sibling paper, [**25**], Rafi took this condition on foliations and translated it into the context of the curve complex. He proves that $\mathcal{G}$ enters the thin part of $\mathcal{T}(S)$ of a subsurface $Y \subset S$ if and only if the curves which constitute ∂Y are short along $\mathcal{G}$, which happens if and only if Y is filled by subsurfaces to whose curve complexes the vertical and horizontal foliations have sufficiently large projections. In addition, he adapted the Masur–Minsky coarse distance formula for $\mathcal{MCG}(S)$ to obtain a coarse distance formula for $\mathcal{T}(S)$ with the Teichmüller metric (Theorem 2.10; Figure 1).

The outline of the paper is as follows: In Section 2, we give the background necessary for the paper; in Section 3, we show how to build $\mathcal{AM}(S)$ from $\mathcal{M}(S)$; in Section 4, we define augmented hierarchies, and show how to translate most of [**20**] to our setting; in Section 5, we explain how to build augmented hierarchy paths; in Section 6, we derive a distance formula for $\mathcal{AM}(S)$, and prove that augmented hierarchy paths are uniform quasigeodesics; in Section 7,

$$\begin{tikzcd}
(\mathcal{T}(S), d_T) \arrow[r, "\text{Theorem 1.1}"] \arrow[d] & \mathcal{AM}(S) \arrow[d] \\
\mathcal{MCG}(S) \arrow[r, "\text{Masur-Minsky } [\textbf{MM00}]"] \arrow[d] & \mathcal{M}(S) \arrow[d] \\
(\mathcal{T}(S), d_{WP}) \arrow[r, "\text{Brock } [\textbf{Br03}]"] \arrow[d] & \mathcal{P}(S) \arrow[d] \\
\mathcal{T}^{el}(S) \arrow[r, "\text{Masur-Minsky } [\textbf{MM99}]"] & \mathcal{C}(S)
\end{tikzcd}$$

FIGURE 1. *The above figure represents a flow of ideas: the vertical arrows indicate a reduction of complexity, while all horizontal arrows are $\mathcal{MCG}(S)$-equivariant quasiisometries.*

we prove that $\mathcal{AM}(S)$ and $(\mathcal{T}(S), d_T)$ are quasiisometric; finally, in the appendix, we prove structural results about hierarchies which may be of interest to the experts.

2. *Preliminaries*

For the remainder of the paper, let S be a connected, orientable surface of finite type with negative Euler characteristic.

In this section, we recall from [**20**] the basic construction of the marking complex for a surface of finite type, $\mathcal{M}(S)$. We then briefly explain Rafi's combinatorial model [**25**] for Teichmüller space in the Teichmüller metric, $(\mathcal{T}(S), d_T)$. Finally, we introduce the notion of a combinatorial horoball from [**14**].

2.1. *Notation*

To simplify the exposition, we adopt some standard notation from coarse geometry. Given a pair of constants, $C_1, C_2 \geqslant 0$, and a pair of quantities, A and B, we write $A \asymp_{(C_1,C_2)} B$ or simply $A \asymp B$ if

$$\frac{1}{C_1} \cdot A - C_2 \leqslant B \leqslant C_1 \cdot A + C_2.$$

In this paper, any such constants C_1 and C_2 involved in a coarse equality depend on the topology of S.

2.2. *Curve complexes and subsurface projections*

The *complex of curves* of S, denoted by $\mathcal{C}(S)$, is a simplicial complex whose simplices consist of disjoint collections of isotopy classes of simple closed curves on S. In the case where S is a once-punctured torus or four-holed sphere, minimal intersection replaces disjointness as the adjacency relation. For Y_α, an annulus in S with core curve α, $\mathcal{C}(Y_\alpha) = \mathcal{C}(\alpha)$ is the simplicial complex with vertices consisting of paths between the two boundary components of the metric compactification, $\overline{Y_\alpha}$, of $\widetilde{Y}_\alpha$, the cover of S corresponding to Y_α, up to homotopy relative to fixing the endpoints on the boundary; two paths are connected by an edge if they have disjoint interiors.

We will be considering only the 1-skeleton of $\mathcal{C}(S)$ with its path metric. Endowed with this metric, we have a remarkable theorem of Masur and Minsky [**19**].

Theorem 2.1. *The curve complex $\mathcal{C}(S)$ is infinite-diameter and Gromov hyperbolic.*

The curve complex is locally infinite, but the links of vertices are often (products of) Gromov hyperbolic graphs, which gives us a substantial amount of control over the global geometry of $\mathcal{C}(S)$, via the hierarchy machinery in [**20**].

Consider a curve $\alpha \in \mathcal{C}(S)$. Then the link of α is $\mathcal{C}(S \setminus \alpha)$, where $\mathcal{C}(S \setminus \alpha)$ is the join $\mathcal{C}(S_1) * \mathcal{C}(S_2)$ if α is separating and $S \setminus \alpha = S_1 \coprod S_2$. More generally, if $Y \subset S$ is any proper subsurface, then $\mathcal{C}(Y)$ lives in the 1-neighborhood of $\partial Y \subset \mathcal{C}(S)$.

We are often interested in understanding the combinatorial relationship between two curves or simplices of $\mathcal{C}(S)$ from the perspective of $\mathcal{C}(Y)$ for some subsurface $Y \subset S$. Let $\alpha \subset \mathcal{C}(S)$ be any simplex and let $Y \subset S$ be any subsurface of S which is not a pair of pants. The *subsurface projection* of α to Y is the canonical completion of the arcs in $\alpha \cap Y$ along the boundary of a regular neighborhood of $\alpha \cap Y$ and ∂Y to curves in Y. We denote this projection by $\pi_Y(\alpha)$ and remark that it is a simplex in $\mathcal{C}(Y)$. If Y_γ is an annulus with core γ and α intersects γ transversely, then $\pi_\gamma(\alpha)$ is the finite, diameter-1 set of lifts of α to $\widetilde{Y}_\gamma$ which connect the two boundary components of $\overline{Y}_\gamma$; see [**20**, Section 2] for more details.

For any two simplices $\alpha, \beta \subset \mathcal{C}(S)$ and subsurface $Y \subset S$, we use the shorthand $d_Y(\alpha, \beta) = d_Y(\pi_Y(\alpha), \pi_Y(\beta))$.

Subsurface projections are essential objects in the Masur–Minsky hierarchy machinery. One of the main outputs of that machinery is the distance formula for $\mathcal{M}(S)$, Theorem 2.7 below.

2.3. *Marking complexes*

A *marking*, μ, on a surface S is a collection of *transverse pairs*, (α, t_α), where the α form a simplex in $\mathcal{C}(S$, called the *base* of μ, denoted by $\mathrm{base}(\mu)$, and each t_α is a diameter-1 set of vertices in the annular complex $\mathcal{C}(\alpha)$ (see [**20**, Subsection 2.4]), called the set of *transversals*. We say a marking μ is *complete* if $\mathrm{base}(\mu)$ is a pants decomposition of S, and *clean*, if the only base curve each transversal t_α intersects is its paired base curve, α.

We remark that, in any complete clean marking, each transversal intersects either one or two other transversals. Indeed, since the base curves form a pants decomposition, one can decompose S into a collection of pairs of pants, where the base curves form the cuffs and the transverse curves are cut into essential arcs in the pairs of pants. In each pair of pants, each transverse arc must intersect exactly one other transverse arc. In the case that α is two cuffs in one pair of pants (that is, α and t_α fill a one-holed torus), t_α intersects only one other transverse curve; otherwise, each transverse curve intersects two others.

The *marking complex* of S, denoted by $\mathcal{M}(S)$, is a graph whose vertices are complete clean markings, and two markings are connected by an edge if they can be related by one of two types of *elementary moves*, called *twists* and *flips*, which we define now.

Given a marking μ and a pair (α, t_α) in μ, a *twist move* around α involves replacing μ with $T_\alpha(\mu)$, where T_α is a Dehn twist or half-twist around α, depending on whether $\alpha \cup t_\alpha$ fills a once-punctured torus or a four-holed sphere, respectively. By construction, t_α is the only curve in μ which intersects α, so this reduces to $(\alpha, t_\alpha) \mapsto (\alpha, T_\alpha(t_\alpha))$.

Given a pair (α, t_α), a *flip move* performed at α involves a flip $(\alpha, t_\alpha) \mapsto (t_\alpha, \alpha)$ and some extra changes to preserve cleanliness, which we now explain. As noted above, each transverse curve intersects (either one or two) others, so now that a transverse curve has become a base curve, at least one other transverse pair has been made unclean. In [**20**, Lemma 2.4], Masur and Minsky show that by choosing replacement transversals to minimize distance in the annular curve complexes of their bases, one has a finite number of possible new transversals which are all uniformly close to each other. The purpose of this cleaning is to preserve the twisting data around α, while allowing for future flip moves to occur without the resulting base sets failing to be pants decompositions; see Figure 2.

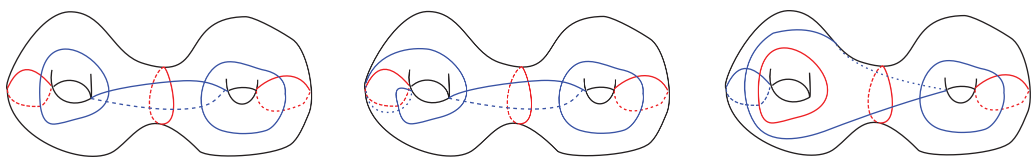

FIGURE 2 (colour online). (a) *A marking on a genus two surface, where (in red) consists of the central separating curve and the outer curves which pass through the genera, and the other three curves are the transversals (in blue);* (b) μ *after a twist move around the left base curve;* (c) μ *after a flip move at the left transverse pair.*

In the rest of the paper, we assume that all markings are clean and complete.

DEFINITION 2.2 (Subsurface projections for markings). We will be interested in subsurface projections for markings. For any $\mu \in \mathcal{M}(S)$ and $Y \subset S$, any subsurface which is not an annulus whose core is in base(μ), we define the *subsurface projection of* μ *to* $\mathcal{C}(Y)$ by $\pi_Y(\mu) = \pi_Y(\text{base}(\mu))$. In the case that Y is an annulus with core $\alpha \in \text{base}(\mu)$ with transversal t_α, then $\pi_Y(\mu) = t_\alpha$.

We now define the projection of a marking on S to a marking on a subsurface.

DEFINITION 2.3 (Projections of markings to markings on subsurfaces). Let $\mu \in \mathcal{M}(S)$ and $Y \subset S$ be any subsurface. We build $\pi_{\mathcal{M}(Y)}(\mu)$ inductively as follows. Choose a curve $\alpha_1 \in \pi_Y(\mu)$; then build a pants decomposition on Y by choosing $\alpha_i \in \pi_{Y \setminus \bigcup_{j=1}^{i-1} \alpha_j}(\mu)$. From this pants decomposition, build a marking on Y by choosing transverse pairs $(\alpha_i, \pi_{\alpha_i}(\mu))$. We define $\pi_{\mathcal{M}(Y)}(\mu) \subset \mathcal{M}(Y)$ to be the collection of all markings resulting from varying the choices of the α_i.

Lemma 2.4 in [20] and Lemma 6.1 of [1] show that the freedom in this process builds a bounded diameter subset of $\mathcal{M}(Y)$. We remark however that if $\partial Y \subset \text{base}(\mu)$, then $\pi_{\mathcal{M}(Y)}(\mu)$ is a unique point in $\mathcal{M}(Y)$, since every curve in base(μ) either projects to itself in $\mathcal{C}(Y)$ or has an empty projection.

REMARK 2.4. The process of constructing $\pi_{\mathcal{M}(Y)}(\mu)$ preserves any curve $\alpha \in \text{base}(\mu)$ which happens to lie in Y, for $\alpha \in \pi_Y(\mu)$ and π_Y preserves disjointness. Otherwise, we could have chosen to build $\pi_{\mathcal{M}(Y)}(\mu)$ by first preferentially choosing curves in base(μ) which lie in Y.

2.4. *Hierarchies, large links, and the Masur–Minsky distance formula*

Since a substantial portion of this paper is spent adapting the Masur–Minsky machinery to Teichmüller space, we now only briefly outline the features of the Masur–Minsky hierarchies. The main references for the hierarchy theory are [20, 23], and we will point the reader to the corresponding sections when possible; the initial exposition begins in [20, Section 4]. See also the theses of Tao [27] and Behrstock [1] for nice introductions to the theory. Our treatment begins in Section 4.

Given any two markings $\mu_1, \mu_2 \in \mathcal{M}(S)$, a *hierarchy* H between μ_1 and μ_2 is family of special geodesics $g_Y \subset \mathcal{C}(Y)$ with partial markings associated, denoted by $\mathbf{I}(g_Y)$ and $\mathbf{T}(g_Y)$. Each such geodesic is supported on a distinct subsurface $Y \subset S$, such that the geodesics satisfy a number of subordinancy relations among the g_Y determined by the associated partial markings; see [20, Subsection 4.1]. Any such hierarchy H can be used to build a uniform quasigeodesic between μ and η in $\mathcal{M}(S)$, called a *hierarchy path*.

Given any pair of markings $\mu_1, \mu_2 \in \mathcal{M}(S)$, we say that a subsurface $Y \subset S$ is a *K-large link* for μ_1 and μ_2 if $d_Y(\mu_1, \mu_2) > K$. Masur and Minsky [**20**, Lemma 6.12] tells us large links are the main building blocks of hierarchy paths.

LEMMA 2.5 ([**20**, Lemma 6.12]). *There exists a $K > 0$ such that, for any $\mu_1, \mu_2 \in \mathcal{M}(S)$ and subsurface $Y \subset S$ such that $d_Y(\mu_1, \mu_2) > K$, then Y supports a geodesic $g_Y \in H$ for any hierarchy H between μ_1 and μ_2.*

REMARK 2.6 (Large link). The intuition behind the term large link is as follows: If $Y \subset S$ is a large link for μ_1, μ_2, then we know from Lemma 2.5 that Y supports some geodesic $g_Y \in H$; moreover, Y will necessarily appear as the component of some $Z \setminus \alpha$, where $Z \subset S$ is a subsurface supporting a geodesic $g_Z \in H$ and $\alpha \in g_Z$. While the length of g_Y in $\mathcal{C}(Y)$ is $d_Y(\mu_1, \mu_2) > K$, g_Y lives in the link of $\alpha \in g_Z$ as a path in $\mathcal{C}(Z)$, and hence the link of α is large from the viewpoint of μ_1 and μ_2.

One of the main results of the hierarchy machinery is the inspirational Masur–Minsky distance formula for $\mathcal{M}(S)$, which says that the $\mathcal{M}(S)$-distance between markings is coarsely the sum of their large links.

THEOREM 2.7 ($\mathcal{M}(S)$ distance formula; [**20**, Theorem 6.12]). *For $K > 0$ as in Lemma 2.5 and any $k > K$, there are $E_1, E_2 > 0$, such that, for any $\mu_1, \mu_2 \in \mathcal{M}(S)$,*

$$d_{\mathcal{M}(S)}(\mu_1, \mu_2) \asymp_{(E_1,E_2)} \sum_{d_Y(\mu_1,\mu_2)>k} d_Y(\mu_1, \mu_2).$$

2.5. *The thick part and Minsky's product regions*

One of the main corollaries to the hyperbolicity of $\mathcal{C}(S)$ is [**19**, Theorem 1.2], which states that the electrification of $(\mathcal{T}(S), d_T)$ is quasiisometric to $\mathcal{C}(S)$. In contrast, Minsky showed in [**22**, Theorem 6.1] that the thin regions of $(\mathcal{T}(S), d_T)$, where at least one curve is short, are quasiisometric to a product space with its sup metric.

Let $\gamma = \gamma_1, \ldots, \gamma_n$ be a simplex in $\mathcal{C}(S)$, and let $Thin_\epsilon(S, \gamma) = \{\sigma \in \mathcal{T}(S) \,|\, l_\sigma(\gamma_i) \leqslant \epsilon\}$, where $l_\sigma(\gamma_i)$ is the hyperbolic length of γ_i in σ for each i. Let

$$\mathcal{T}_\gamma = \mathcal{T}(S \setminus \gamma) \times \prod_{\gamma_i \in \gamma} \mathbb{H}_{\gamma_i} \tag{2.1}$$

be endowed with the sup metric, where $S \setminus \gamma$ is a disjoint union of punctured surfaces and each $\mathbb{H}_{\gamma_i}$ is a *horodisk*, that is, a copy of the upper half-plane model of $\mathbb{H}^2$ with imaginary part at least 1.

THEOREM 2.8 (Product regions; [**22**, Theorem 6.1]). *The Fenchel–Nielsen coordinates on $\mathcal{T}(S)$ give rise to a natural homeomorphism $\Pi : \mathcal{T}(S) \to \mathcal{T}_\gamma$, and, for $\epsilon > 0$ sufficiently small, this homeomorphism restricted to* $\mathrm{Thin}_\epsilon(S, \gamma)$ *distorts distances by a bounded additive amount.*

In what follows, fix $\epsilon > 0$ to be sufficiently small so that Theorem 2.8 holds. When we say that a curve α is *short* for some $\sigma \in \mathcal{T}(S)$, we mean that $l_\sigma(\alpha) < \epsilon$.

REMARK 2.9. Up to quasiisometry, we may take the sup or product metric on the product space in (2.1), though Minsky's version with the sup metric is finer and results in only an additive error.

2.6. *Rafi's combinatorial model*

The main result of [**25**] is an adaptation of the machinery in [**20**] to the setting of $(\mathcal{T}(S), d_T)$. In particular, Rafi obtains a distance estimate in [**25**, Theorem 6.1] analogous to the Masur–Minsky formula (Theorem 2.7 above), restated below in Theorem 2.10.

Given $\sigma \in \mathcal{T}(S)$, a *shortest marking* $\mu_\sigma \in \mathcal{M}(S)$ for σ is a marking inductively built by choosing a shortest curve in $\alpha_1 \in \mathcal{C}(S)$ on σ with respect to extremal length, Ext_σ, then choosing a shortest curve $\alpha_2 \in \mathcal{C}(S \setminus \alpha_1)$, and so on, until one has arrived at a shortest pants decomposition of S. One completes this to a shortest marking by choosing shortest curves β_i, which intersect α_i, but not α_j for $j \neq i$. The result is a complete, clean marking, of which there are finitely many by [**20**, Lemma 2.4]. We note that the collection of curves which are shorter in σ than the constant ϵ in Minsky's Theorem 2.8, form a simplex in $\mathcal{C}(S)$ by the Collar Lemma. Thus in the case that $\sigma \in Thin_\gamma$ for some simplex $\gamma \subset \mathcal{C}(S)$, we necessarily have $\gamma \subset \mathrm{base}(\mu_\sigma)$.

Theorem 2.10 (Rafi's formula; [**25**, Theorem 6.1]). *Let $\epsilon > 0$ be as in Theorem 2.8. There exists $k > 0$ such that the following holds.*

Let $\sigma_1, \sigma_2 \in \mathcal{T}(S)$, define Λ to be the set of curves short in both σ_1 and σ_2, and define Λ_i to be the set of curves short in σ_1 and not in Λ. Let μ_i be the shortest marking for σ_i. Then

$$d_{\mathcal{T}}(\sigma_1, \sigma_2) \asymp \sum_Y [d_Y(\mu_1, \mu_2)]_k + \sum_{\alpha \notin \Lambda} \log[d_\alpha(\mu_1, \mu_2)]_k + \max_{\alpha \in \Lambda} d_{\mathbb{H}_\alpha}(\sigma_1, \sigma_2) + \max_{\substack{\alpha \in \Lambda_i \\ i=1,2}} \log \frac{1}{l_{\sigma_i}(\alpha)}.$$

One of the main products of this paper, Theorem 7.14, is an independent, combinatorial proof of Rafi's distance formula.

2.7. *Bers pants decompositions*

Our augmented markings are markings with some length data. When we associate a point in $\mathcal{T}(S)$ to an augmented marking, it will be important that the extremal lengths of the curves we choose for the marking are uniformly bounded. We will not use the greedy algorithm used to build the shortest markings for Rafi's Theorem 2.10. Recall the following theorem of Bers.

Theorem 2.11 (Bers). *There is a constant $L > 0$, depending only on the topology of S, such that, for any point $\sigma \in \mathcal{T}(S)$, there is a pants decomposition P_σ with $l_\sigma(\alpha) < L$ for each $\alpha \in P_X$.*

For any $X \in \mathcal{T}(S)$, any $P_X \in \mathcal{P}(S)$ as in Theorem 2.11 is called a *Bers pants decomposition*.

Next, recall the well-known fact due to Maskit [**18**], which relates hyperbolic and extremal length.

Lemma 2.12. *Let $\sigma \in \mathcal{T}(S)$, $\alpha \in \mathcal{C}(S)$, and $C > 0$. Then $l_\sigma(\alpha) \asymp 1$ if and only if $\mathrm{Ext}_\sigma(\alpha) \asymp 1$.*

The following lemma is a consequence of the Collar Lemma.

Lemma 2.13. *There exist constants $\epsilon_0, L_0 > 0$, depending only on S such that the following holds. Let $\sigma \in \mathcal{T}(S)$ and let P_σ be any Bers pants decomposition for σ. Then the following conditions are satisfied.*

(i) *For any $\alpha \in \mathcal{C}(S)$, if $l_\sigma(\alpha) < \epsilon_0$, then $\alpha \in P_\sigma$.*
(ii) *For any $\beta \in P_\sigma$, $\mathrm{Ext}_\sigma(\beta) < L_0$.*

Proof. For (1), we can choose ϵ_0 small enough so that if γ intersects α, where $l_\sigma(\alpha) < \epsilon_0$, then $l_\sigma(\gamma) > L$ by the Collar Lemma. For (2), the Collar Lemma states that there is a regular neighborhood of β on σ, with diameter depending only on $l_\sigma(\beta)$, which is an embedded annulus. The reciprocal of this diameter is thus both an upper bound for $\mathrm{Ext}_\sigma(\beta)$ and bounded above by Lemma 2.12, completing the proof. □

For the rest of the paper, fix $\epsilon_0 > 0$ sufficiently small, satisfying both Lemma 2.13 and Theorem 2.8.

2.8. *Combinatorial horoballs*

Combinatorial horoballs were introduced by Groves and Manning in [**14**] in the context of relatively hyperbolic groups; see [**10**] for an earlier, similar construction. In particular, suppose that G is a finitely generated group and $\mathcal{P} = \{P_1, \ldots, P_n\}$ is a finite collection of finitely generated subgroups of G. Among other equivalences, in [**14**, Theorem 3.25] they showed that the augmentation of the Cayley graph of G by combinatorial horoballs along the subgroups in $\mathcal{P}$ is hyperbolic if and only if G is relatively hyperbolic to $\mathcal{P}$ in the sense of Gromov.

While $\mathcal{MCG}(S)$ is not relatively hyperbolic to any family of subgroups [**2**], the process of adding efficient paths to the marking complex via combinatorial horoballs to build the augmented marking complex is reminiscent of and indeed inspired by the relatively hyperbolic construction. We use combinatorial horoballs to model the hyperbolic upper half-planes which appear in the product structure of the thin parts discovered by Minsky [**22**] in Theorem 2.8. We fully explain the construction of $\mathcal{AM}(S)$ in the next section.

Definition 2.14 (Combinatorial horoball). Let X be any simplicial complex. The *combinatorial horoball based on* X, $\mathcal{H}(X)$, is the 1-complex with vertices $\mathcal{H}(X)^{(0)} = X^{(0)} \times (\{0\} \cup \mathbb{N})$ and edges as follows.

(i) If $x, y \in X^{(0)}$ and $n \in \{0\} \cup \mathbb{N}$ such that $0 < d_X(x, y) \leqslant e^n$, then (x, n) and (y, n) are connected by an edge in $\mathcal{H}(X)$.

(ii) If $x \in X^{(0)}$ and $n \in \{0\} \cup \mathbb{N}$, then (x, n) is connected to $(x, n+1)$ by an edge.

The metric on $\mathcal{H}(X)$ is the path metric, where each edge is isometric to $[0, 1]$.

Remark 2.15. The simplicial complex X sits inside of $\mathcal{H}(X)$ as the full subgraph containing the vertices $X^{(0)} \times \{0\}$.

As with horoballs in $\mathbb{H}^n$, combinatorial horoballs are uniformly hyperbolic.

Theorem 2.16 ([**14**, Theorem 3.8]). *Let X be any simplicial complex. Then $\mathcal{H}(X)$ is δ-hyperbolic where δ is independent of X.*

The following is a usual fact from Groff [**13**, Lemma 6.2].

Lemma 2.17. *Let $q : A \to B$ be a (k, c)-quasiisometry of graphs. Then there exists a $(1, C)$-quasiisometry $\hat{q} : \mathcal{H}(A) \to \mathcal{H}(B)$, where C depends only on k and c.*

We need to understand efficient paths in combinatorial horoballs. Fortunately, they have a nice description from [**14**, Lemma 3.10].

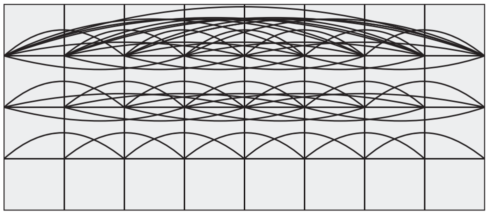

FIGURE 3. *A busy* (4×8)*-slice of the base of a combinatorial horoball over* $\mathbb{Z}$*; every edge has length* 1*. Note that at height* 2*, each vertex is connected to half the others by edges, while all vertices are connected by edges at height* 3*.*

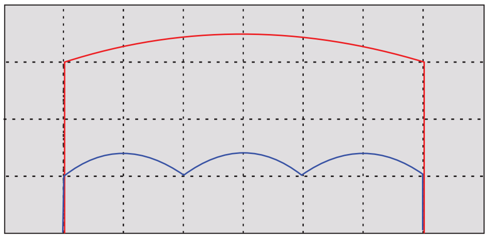

FIGURE 4 (colour online). *Two paths between a pair of points in a combinatorial horoball: The top path is a preferred path, while the bottom path is a geodesic.*

LEMMA 2.18 ([**14**, Lemma 3.10]). *Let* $\mathcal{H}(X)$ *be a combinatorial horoball and* $x, y \in \mathcal{H}(X)$ *be distinct vertices. Then there is a uniform quasigeodesic* $\gamma(x,y) = \gamma(y,x)$ *between* x *and* y*, which consists of at most two vertical segments and a single horizontal segment of length at most* 3*.*

Moreover, any other geodesic between x *and* y *is Hausdorff distance at most* 4 *from this quasigeodesic, and no geodesic can have a horizontal segment of length greater than* 4*.*

Following [**14**, Subsection 5.1], we define preferred paths for $\mathcal{H}(X)$.

Suppose that $x, y \in X$ have $d_X(x,y) = C$. For any $(x,a), (y,b) \in \mathcal{H}(X)$, consider the path between these two points which consists of (at most) three segments: a vertical segment from (x,a) to $(x, \lceil \ln C \rceil)$, a horizontal segment of one edge from $(y, \lceil \ln C \rceil)$, and another vertical segment from $(y, \lceil \ln C \rceil)$ to (y,b). In the case that a or $b \geqslant \ln C$, then the respective vertical segment is not included, and the horizontal segment connects at either height a or b, depending on whether or not $a \geqslant b$.

These paths are not geodesics (which are similar, but will differ slightly in vertical height depending on the divisibility of C), but they are uniform quasigeodesics which are a uniformly bounded distance from geodesics, with the bound independent of X. This can be seen from the easily verified fact that no geodesic can contain a horizontal segment of length greater than 5 (see Figure 3 in the [**14**, proof of Lemma 3.11]). Because they are easy to define, these are the preferred paths through horoballs we consider in what follows. It is obvious from their definition that they are unique; see Figure 4.

3. *Construction of* $\mathcal{AM}(S)$

The main idea of the construction of $\mathcal{AM}(S)$ is to model the product regions discovered by Minsky [**22**] using $\mathcal{M}(S)$ as the thick part. We begin by showing a combinatorial horoball over an orbit of a Dehn twist or half-twist in $\mathcal{M}(S)$ is quasiisometric to a horodisk. We then define $\mathcal{AM}(S)$ as a graph and make some observations about its structure. We finish the section by defining the maps identifying $\mathcal{AM}(S)$ with $\mathcal{T}(S)$, and prove some basic facts about the identification.

3.1. *The horoballs $\mathcal{H}_\alpha$ are quasiisometric to horodisks*

Let $\mathcal{H}_{(\alpha,t_\alpha)}$ be the combinatorial horoball over the orbit of the action of $\langle T_\alpha\rangle$ on μ, where μ contains a transverse pair (α,t_α). A typical point in $\mathcal{H}_{(\alpha,t_\alpha)}$ is of the form $(\alpha,T^k_\alpha(t_\alpha),n)$, where $T^k_\alpha(t_\alpha)$ records the horizontal position, n records the vertical position, and α and t_α identify the particular horoball. When the context is clear, we write $(\alpha,T^k_\alpha(t_\alpha),n)=(k,n)$. We also frequently suppress the transverse curve when referring to a horoball, and simply write $\mathcal{H}_\alpha$ when the context is clear.

We begin this section with an elementary proof of the fact that horodisks are quasiisometric to combinatorial horoballs over orbits of Dehn twists or half-twists. In order to do this, we use a set of criteria for a map to be a quasiisometry from the lemma in [**10**, Subsection 4.2].

LEMMA 3.1. *Let X and Y be spaces with path metrics. In order for $\phi : X\to Y$ to be a quasiisometry, it suffices that*

(i) *for some $L>0$, $Y\subset N_L(\phi(X))$;*
(ii) *for some $K>0$ and for all $x_1,x_2\in X$, $d_Y(\phi(x_1),\phi(x_2))\leqslant K\cdot d_X(x_1,x_2)$; and*
(iii) *for each $M>0$ there exists an $N>0$ such that if $d_X(x_1,x_2)>N$, then $d_Y(\phi(x_1),\phi(x_2))>M$.*

PROPOSITION 3.2 (Horoballs are quasiisometric to horodisks). *Let $\mu\in\mathcal{M}(S)$, (α,t_α) a transverse pair in μ, and $\mathcal{H}_\alpha$ be the combinatorial horoball over the orbit of the action of $\langle T_\alpha\rangle$ on μ. Then $\mathcal{H}_\alpha$ with the path metric is quasi-isometric to a horodisk with the Poincaré metric.*

Proof of Proposition 3.2. Let Δ be the standard horodisk with the Poincaré metric. Define a map $\phi:\mathcal{H}_\alpha\to\Delta$ by $\phi(\alpha,T^k_\alpha(t_\alpha),n)=\phi(k,n)=(k,e^n)$. We verify that ϕ satisfies the conditions from Lemma 3.1.

To see that $\phi(\mathcal{H}_\alpha)$ is quasidense in Δ, and thus satisfies condition (i), observe that $\phi(\mathcal{H}_\alpha)$ is all the points of the form (n,e^k), where $n,k\in\mathbb{Z}_{\geqslant 0}$. Since the Δ-distance between two horizontally adjacent vertices in $\phi(\mathcal{H}_\alpha)$ is uniformly bounded by the distance between two vertices at height 1, every point in Δ is at most distance 1 from a vertical geodesic line in $\phi(\mathcal{H}_\alpha)$. Similarly, the distance between two vertically adjacent vertices in $\phi(\mathcal{H}_\alpha)$ is bounded by $(e-1)/e$. Thus $\phi(\mathcal{H}_\alpha)$ is quasidense in Δ.

We now verify condition (ii) on endpoints of edges of $\mathcal{H}_\alpha$. Vertical edges are geodesics in $\mathcal{H}_\alpha$ and ϕ sends them to vertical segments, which are geodesics of the same length in Δ. Similarly, a horizontal edge in $\mathcal{H}_\alpha$, connecting (k_1,n) and (k_2,n), where $|k_1-k_2|<e^n$, is a geodesic of length 1. A calculation verifies that the $d_\Delta((k_1,e^n),(k_2,e^n))$ is bounded by $1/\sqrt{2}$, confirming condition (ii).

Finally, we check condition (iii). Suppose that we have $x_1=(k_1,n_1),x_2=(k_2,n_2)\in\mathcal{H}_\alpha$ such that $d_\Delta((k_1,e^{n_1}),(k_2,e^{n_2}))$ is bounded. We claim that implies $|k_1-k_2|$ and $|n_1-n_2|$ are bounded. From this, it follows immediately that $d_{\mathcal{H}_\alpha}((k_1,n_1),(k_2,n_2))$ is bounded, confirming condition (iii) for the vertices.

Now we check condition (iii) for points in the interior of the edges. Assume that at least one of $|k_1-k_2|,|n_1-n_2|$ is large, for a contradiction. As noted above, ϕ sends vertical geodesics in $\mathcal{H}_\alpha$ to vertical geodesics in Δ of the same length, and hence if $k_1=k_2$, then $d_{\mathcal{H}_\alpha}(x_1,x_2)=d_\Delta(\phi(x_1),\phi(x_2))$, so we may assume $k_1\neq k_2$. Without loss of generality, assume that $k_1<k_2$ and $n_1\leqslant n_2$. Consider the Δ-geodesic triangle $\triangledown$ with vertices $\bar a=[(k_1,e^{n_1}),(k_1,e^{n_2})]$, $\bar b=[(k_1,e^{n_2}),(k_2,e^{n_2})]$, $\bar c=[(k_1,e^{n_1}),(k_2,e^{n_2})]$; we note that $|\bar c|_\Delta=d_\Delta(\phi(x_1),\phi(x_2))$.

Since we are assuming that $|\bar c|$ is bounded, our assumption that one of $|k_1-k_2|$ or $|n_1-n_2|$ is large, implies that one of $|\bar a|$ or $|\bar b|$ is large. It follows immediately the triangle inequality that both $|\bar a|$ and $|\bar b|$ are large. By δ-hyperbolicity of Δ, $\triangledown$ is δ-thin. Note that the angle in

$\bigtriangledown$ at the vertex (k_1, e^{n_2}), where $\bar{a}$ and $\bar{b}$ meet is bigger than $\pi/2$. If we parameterize $\bar{a}$ and $\bar{b}$ moving away from (k_1, e^{n_2}) by $f_{\bar{a}} : [0, |\bar{a}|] \to \Delta$ and $f_{\bar{b}} : [0, |\bar{b}|] \to \Delta$, then $d_\Delta(f_{\bar{a}}(t), f_{\bar{b}}(t)) > \delta$ for $t > \delta$. Thus δ-thinness of $\bigtriangledown$ implies that $\bar{c}$ must be δ-close to $\bar{a}$ and $\bar{b}$ for almost their entire lengths. Since they were long, it implies that $\bar{c}$ must have been long, which is a contradiction. $\square$

3.2. *Building $\mathcal{AM}(S)$ from $\mathcal{M}(S)$*

We are now ready to define the augmented marking complex for a surface, denoted by $\mathcal{AM}(S)$. $\mathcal{AM}(S)$ is a simplicial 1-complex with vertices and edges as follows.

A vertex $\widetilde{\mu} \in \mathcal{AM}^{(0)}(S)$, called an *augmented marking*, is a complete clean marking, $\pi_{\mathcal{M}(S)}(\widetilde{\mu}) = \mu \in \mathcal{M}(S)$ along with a collection of lengths for the curves in $\text{base}(\mu) = \{\alpha_1, \ldots, \alpha_n\}$:

$$\widetilde{\mu} = (\mu, D_{\alpha_1}(\widetilde{\mu}), \ldots, D_{\alpha_n}(\widetilde{\mu})),$$

where the $D_{\alpha_i}(\mu)$ are nonnegative integers. The $D_{\alpha_i}(\widetilde{\mu})$ are called the *length data* of $\widetilde{\mu}$. When the context is clear, we shorten this to D_α. We also write $(\alpha, t_\alpha, D_\alpha) \in \widetilde{\mu}$ if $\alpha \in \text{base}(\widetilde{\mu})$ with transverse curve t_α and length D_α.

REMARK 3.3 (Thick and thin). The integer D_{α_i} coarsely stands in for how short α_i is in a given augmented marking, in terms of extremal (or hyperbolic) length, with D_{α_i} positive, implying α_i is short; this analogy is made explicit in the definition of the map $G : \mathcal{AM}(S) \to \mathcal{T}(S)$ in Subsection 7.3. When $D_{\alpha_i}(\widetilde{\mu}) = 0$ for all $\alpha_i \in \text{base}(\mu)$, we say that $\widetilde{\mu}$ is *in the thick part of* $\mathcal{AM}(S)$. Similarly, if $D_{\alpha_i}(\widetilde{\mu}) > 0$, then we say α_i is *short* in $\widetilde{\mu}$ and $\widetilde{\mu}$ is in the *α_i-thin part* of $\mathcal{AM}(S)$.

More generally, let $\rho \subset \mathcal{C}(S)$ be a simplex. We say that $\widetilde{\mu} \in \mathcal{AM}(S)$ is in the *ρ-thin part* of $\mathcal{AM}(S)$ if $D_\alpha(\widetilde{\mu}) > 0$ for each $\alpha \in \rho$. If, in addition, $D_\beta(\widetilde{\mu}) = 0$ for all $\beta \in \mathcal{C}(S \setminus \rho)$, then we say that $\widetilde{\mu}$ is *thick relative to ρ*.

There are three types of edges in $\mathcal{AM}^{(1)}(S)$. The first type is the elementary *flip move* from $\mathcal{M}(S)$. The second type is a *twist move*, which comes from bundles of elementary twist moves from $\mathcal{M}(S)$ and corresponds to a horizontal edge in a combinatorial horoball. The last type is a *vertical move*, which involves adjusting the length data and corresponds to a vertical edge in a combinatorial horoball. We connect two augmented markings $\widetilde{\mu}_1, \widetilde{\mu}_2 \in \mathcal{AM}^{(0)}(S)$ by an edge in each of the following cases.

(1) *Flip moves*: If $\mu_1, \mu_2 \in \mathcal{M}(S)$ differ by a flip move at a transverse pairing $(\alpha, t) \mapsto (t, \alpha)$, and if $\widetilde{\mu}_1, \widetilde{\mu}_2$ have the same base curves and length data, with $D_\alpha(\widetilde{\mu}_1) = D_\alpha(\widetilde{\mu}_2) = 0$.

(2) *Twist moves*: If $\alpha \in \text{base}(\mu_1) = \text{base}(\mu_2)$, $D_\alpha(\widetilde{\mu}_1) = D_\alpha(\widetilde{\mu}_2) = k > 0$, and $\widetilde{\mu}_1 = T_\alpha^n \widetilde{\mu}_2$ with $0 < n < e^k$.

(3) *Vertical moves*: If $\mu_1 = \mu_2$ and if $\widetilde{\mu}_1, \widetilde{\mu}_2$ only differ in length data by 1 in one component, say $D_\alpha(\widetilde{\mu}_1) = D_\alpha(\widetilde{\mu}_2) + 1$ and $D_\beta(\widetilde{\mu}_1) = D_\beta(\widetilde{\mu}_2)$ for all $\beta \in \text{base}(\mu_1) \setminus \alpha = \text{base}(\mu_2) \setminus \alpha$.

REMARK 3.4 (No flipping a short curve). If $\widetilde{\mu} \in \mathcal{AM}(S)$, $D_\alpha(\widetilde{\mu}) > 0$, and (α, t) is a transverse pair, then it is not possible, by construction, to perform a flip move $(\alpha, t) \mapsto (t, \alpha)$, for only base curves can be short. This is precisely to guarantee that the Teichmüller distance between the image under the map G of two augmented markings which differ by an elementary move is uniformly bounded; see Lemma 7.10.

Since $\mathcal{M}(S)$ is locally finite and each augmented marking has at most two vertical edges for each base curve, we have the following immediately from the definition.

LEMMA 3.5. *The augmented marking complex $\mathcal{AM}(S)$ is locally finite, but not uniformly locally finite.*

The metric on $\mathcal{AM}(S)$ is the path metric, where each edge is given length 1. We close this subsection with a series of remarks.

REMARK 3.6 ($\mathcal{M}(S) \hookrightarrow \mathcal{AM}(S)$). For any subsurface $Y \subset S$, there is a natural inclusion of $i_Y : \mathcal{M}(Y) \hookrightarrow \mathcal{AM}(Y)$ given by $i_Y(\mu) = (\mu, 0, \ldots, 0)$, and we call this embedded copy of $\mathcal{M}(S)$ the *thick part* of $\mathcal{AM}(Y)$ and points therein *thick points*. In particular, when $Y = S$, we think of $i_S(\mathcal{M}(S)) \subset \mathcal{AM}(S)$ as the thick part of $\mathcal{AM}(S)$. As we will see in Subsection 7.3, $i_S(\mathcal{M}(S))$ can be identified with the thick part of $\mathcal{T}(S)$, justifying our terminology.

REMARK 3.7 (Combinatorial horoballs in $\mathcal{AM}(S)$). Let $\mu \in \mathcal{M}(S)$ and (α, t) a transverse pair in μ. Consider the orbit, $X_\alpha \subset \mathcal{M}(S)$, of μ under $\langle T_\alpha \rangle \leqslant \mathcal{MCG}(S)$, the subgroup generated by the Dehn twist or half-twist about α. Consider the image of X_α in $\mathcal{AM}(S)$, namely $i_S(X_\alpha)$. Then $i_S(X_\alpha)$ lies at the base of the combinatorial horoball $\mathcal{H}_\alpha \subset \mathcal{AM}(S)$.

REMARK 3.8 (Shadows). There is a natural map $\pi_{\mathcal{M}(S)} : \mathcal{AM}(S) \to \mathcal{M}(S)$ defined by $\pi_{\mathcal{M}(S)}(\widetilde{\mu}) = \mu$ for any $\widetilde{\mu} \in \mathcal{AM}(S)$, which we call the *shadow map*. Similarly, any path in $\mathcal{AM}(S)$ shadows a path in $\mathcal{M}(S)$.

REMARK 3.9 (Thin parts and product regions). Let $\rho \subset \mathcal{C}(S)$ be a simplex. If we ignore the technical concerns about cleaning markings after flip moves, then the collection of ρ-thin points in $\mathcal{AM}(S)$, which we call *the ρ-thin part of $\mathcal{AM}(S)$*, coarsely has the structure of the 1-skeleton of $\prod_{\alpha \in \rho} \mathcal{H}_\alpha \times \mathcal{AM}(S \setminus \rho)$ (see Theorem 2.8 for comparison).

4. *Augmented hierarchies*

In this section, we develop the $\mathcal{AM}(S)$-analogue of the Masur–Minsky hierarchy machinery. Informally, an augmented hierarchy will be a hierarchy in which the geodesics in annular curve complexes have been replaced by geodesics in combinatorial horoballs. Much of the work in [**20**] goes through to this setting unchanged, as the role the annular geodesics play in a standard hierarchy almost entirely hinges on the core of the annuli in question.

4.1. *Combinatorial horoballs over annular curve graphs*

We must first replace annular curve graphs with combinatorial horoballs over them. Recall from Subsection 2.8 that any graph admits a combinatorial horoball, that combinatorial horoballs are uniformly hyperbolic (Theorem 2.16), and that the combinatorial horoballs over quasiisometric graphs are quasiisometric (Lemma 2.17).

Following [**20**, Subsection 2.4], we observe that annular curve graphs $\mathcal{C}(\alpha)$ are quasiisometric to $\mathbb{Z}$. For any curve $\alpha \in \mathcal{C}(S)$, choose an arc $\beta_\alpha \in \mathcal{C}(\alpha)$. For $\gamma \in \mathcal{C}(\alpha)$, let $\gamma \cdot \beta$ denote the algebraic intersection number of γ with β. The map $\phi_{\beta_\alpha} : \mathcal{C}(\alpha) \to \mathbb{Z}$, given by $\phi_{\beta_\alpha}(\gamma) = \gamma \cdot \beta$, is a $(1, 2)$-quasiisometry, independent of the choice of β. The map ϕ_{β_α} essentially records twisting around α relative to β.

Lemma 2.17 implies that $\mathcal{H}(\mathcal{C}(\alpha)) = \mathcal{H}(\alpha)$ is uniformly quasiisometric to $\mathcal{H}(\mathbb{Z})$ for each $\alpha \in \mathcal{C}(S)$. Proposition 3.2 gives us the following lemma.

LEMMA 4.1. *For any $\alpha \in \mathcal{C}(S)$, $\mathcal{H}(\alpha)$ is uniformly quasiisometric to a horodisk in $\mathbb{H}^2$.*

Vertices $x \in \mathcal{H}(\alpha)$ are pairs, $x = (t_\alpha, D_\alpha)$, where $t_\alpha \in \mathcal{C}(\alpha)$ and $D_\alpha \in \mathbb{Z}_{\geqslant 0}$.

In what follows, we build augmented hierarchies by replacing geodesics in $\mathcal{C}(\alpha)$ with geodesics in $\mathcal{H}(\alpha)$.

4.2. *Augmented hierarchies defined*

In this subsection, we will define augmented hierarchies, following the lead of [**20**, Sections 4 and 5].

Let $Y \subset S$ be nonannular and $g \in \mathcal{C}(Y)$ be a geodesic $v_1, \ldots, v_n$, where the vertices v_i are possibly simplices. For any $i \geqslant 1$, note that $v_i \cap v_{i+2} \neq \emptyset$ since g is a geodesic. Let $F(v_i \cup v_{i+2})$ be the subsurface of Y which they fill. We say that g is *tight* if $\partial F(v_i \cup v_{i+2}) = v_{i+1}$, for each i and g has associated *initial* and *terminal* augmented markings, $\widetilde{\mathbf{I}}(g)$ and $\widetilde{\mathbf{T}}(g)$, respectively; tight geodesics exist by [**20**, Lemma 4.5]. If Y is an annulus with core α, then we take $\mathcal{C}(Y) = \mathcal{H}(\alpha)$ and we adopt the convention that any geodesic in $\mathcal{H}(\alpha)$ is tight. From now on, we will assume that all such marked geodesics are tight.

Let $Y \subset S$ be a nonannular subsurface and $\widetilde{\mu} \in \mathcal{AM}(S)$ be an augmented marking. The *restriction of* $\widetilde{\mu}$ *to* Y, denoted by $\widetilde{\mu}_{|Y}$, is the set of transverse triples $(\alpha, t_\alpha, D_\alpha)$ in $\widetilde{\mu}$, whose base curve α meets Y essentially. If $Y \subset S$ is an annulus, then we set $\widetilde{\mu}_{|Y} = \pi_{\mathcal{H}(\alpha)}(\widetilde{\mu})$.

Let $X, Y \subset S$ be subsurfaces with X nonannular. Let $g_X \subset \mathcal{C}(X)$ be a geodesic. We say that Y is a *component domain* of g_X if Y is a component of $X \setminus v$ for some $v \in g_X$. Suppose that Y is component domain for the ith vertex of g_X, namely $v_i \in g_X$, $Y \subset X \setminus v_i$. We note that this determines v_i uniquely.

We define the *initial augmented marking of* Y *relative to* g_X to be

$$\widetilde{\mathbf{I}}(Y, g_X) = \begin{cases} v_{i-1} & \text{if } v_i \text{ is not the first vertex of } g_X, \\ \widetilde{\mathbf{I}}(g_X)_{|Y} & \text{if } v_i \text{ is the first vertex of } g_X. \end{cases}$$

Similarly, we define the *terminal augmented marking of* Y *relative to* g_X to be

$$\widetilde{\mathbf{T}}(Y, g_X) = \begin{cases} v_{i+1} & \text{if } v_i \text{ is not the last vertex,} \\ \widetilde{\mathbf{T}}(g_X)_{|Y} & \text{if } v_i \text{ is the last vertex.} \end{cases}$$

We say that a subsurface $Y \subset S$ is *directly backward subordinate* to g_X, and write $g_X \swarrow Y$ if Y is a component domain of g_X and $\widetilde{\mathbf{I}}(Y, g_X) \neq \emptyset$. Similarly, $Y \subset S$ is *directly forward subordinate* to g_Z, written as $Y \searrow g_Z$, if Y is a component domain of g_Z and $\widetilde{\mathbf{T}}(Y, g_Z) \neq \emptyset$. For a tight geodesic $g_Y \subset \mathcal{C}(Y)$, we write $g_X \swarrow g_Y$ if $g_X \swarrow Y$ and $\widetilde{\mathbf{I}}(g_Y) = \widetilde{\mathbf{I}}(Y, g_X)$; similarly, we write $g_Y \searrow g_Z$ if $Y \searrow g_Z$ and $\widetilde{\mathbf{T}}(g_Y) = \widetilde{\mathbf{T}}(Y, g_Z)$.

We can now state the definition of an augmented hierarchy, which is essentially [**20**, Definition 4.4].

DEFINITION 4.2 (Augmented hierarchies). A *hierarchy* between two augmented markings $\widetilde{\mu}, \widetilde{\eta} \in \mathcal{M}(S)$ is a collection of tight geodesics $\widetilde{H}$, satisfying the following.

(H1) There is a distinguished *main geodesic*, $\tilde{g}_{\widetilde{H}} \in \widetilde{H}$ with $D(\tilde{g}_{\widetilde{H}}) = S$, such that $\widetilde{\mathbf{I}}(\tilde{g}_{\widetilde{H}}) = \mu$ and $\widetilde{\mathbf{T}}(\tilde{g}_{\widetilde{H}}) = \eta$.

(H2) Let $\tilde{g}_X, \tilde{g}_Z \in \widetilde{H}$ and $Y \subset S$ such that $\tilde{g}_X \swarrow Y \searrow \tilde{g}_Z$; then there is a unique $\tilde{g}_Y \in \widetilde{H}$ with $\tilde{g}_X \swarrow \tilde{g}_Y \searrow \tilde{g}_Z$.

(H3) For every $\tilde{g}_Y \in \widetilde{H}$ with $\tilde{g}_Y \neq \tilde{g}_{\widetilde{H}}$, there are $\tilde{g}_X, \tilde{g}_Z \in \widetilde{H}$ with $\tilde{g}_X \swarrow \tilde{g}_Y \searrow \tilde{g}_Z$.

4.3. *Augmented hierarchies exist*

The proof of the existence of augmented hierarchies hews closely to the original proof of the existence of hierarchies in [**20**, Theorem 4.6].

Theorem 4.3 (Augmented hierarchies exist). *Given any pair of augmented markings* $\widetilde{\mu}, \widetilde{\eta} \in \mathcal{AM}(S)$, *there exists an augmented hierarchy* $\widetilde{H}$ *with* $\widetilde{\mathbf{I}}(\widetilde{H}) = \widetilde{\mu}$ *and* $\widetilde{\mathbf{T}}(\widetilde{H}) = \widetilde{\eta}$.

Proof. We say that a collection of tight geodesics $\widetilde{H}$ is a *partial augmented hierarchy* if it satisfies conditions (1) and (3), and the uniqueness part of (2) from Definition 4.2, but not necessarily the existence part.

Choose vertices $P \in \{\mathrm{base}(\widetilde{\mu})$ and $Q \in \{\mathrm{base}(\widetilde{\eta})$, and let $\tilde{g}_{\widetilde{H}} \in \mathcal{C}(S)$ be any tight geodesic between them with $\widetilde{\mathbf{I}}(\tilde{g}_{\widetilde{H}}) = \widetilde{\mu}$ and $\widetilde{\mathbf{T}}(\tilde{g}_{\widetilde{H}}) = \widetilde{\eta}$. Then $\widetilde{H}_0 = \{\tilde{g}_{\widetilde{H}}\}$ is a partial augmented hierarchy, and we will construct a finite sequence of partial augmented hierarchies $\widetilde{H}_n$, which terminates in an augmented hierarchy.

We call a triple $(Y, \tilde{b}, \tilde{f})$ with domain Y and $\tilde{b}, \tilde{f} \in \widetilde{H}_n$ an *unutilized configuration* if $\tilde{b} \swarrow Y \searrow \tilde{f}$, but Y does not support a geodesic $\tilde{k}$ in $\widetilde{H}_n$ with $\tilde{b} \swarrow \tilde{k} \searrow \tilde{f}$.

Let $(Y_n, \tilde{b}_n, \tilde{f}_n)$ be any unutilized configuration in $\widetilde{H}_n$. Let $\tilde{g}_{Y_n} \subset \mathcal{C}(Y_n)$ be any tight geodesic with $\widetilde{\mathbf{I}}(\tilde{g}_{Y_n}) = \widetilde{\mathbf{I}}(Y_n, \tilde{b}_n)$ and $\widetilde{\mathbf{T}}(\tilde{g}_{Y_n}) = \widetilde{\mathbf{T}}(Y_n, \tilde{f}_n)$. Then $\tilde{b}_n \swarrow \tilde{g}_{Y_n} \searrow \tilde{f}_n$ and we can take $\widetilde{H}_{n+1} = \widetilde{H} \cup \{\tilde{g}_{Y_n}\}$.

It is easy to see that the number of domains Y of each complexity $\xi(Y) = m$ for $m < \xi(S)$ supporting unutilized triples is nonincreasing as a function of n. Since each step $\widetilde{H}_n \to \widetilde{H}_{n+1}$ eliminates an unutilized domain, the sequence $\widetilde{H}_n$ is finite and the terminal partial augmented hierarchy $\widetilde{H}$ is an augmented hierarchy. □

4.4. *Hierarchies associated to an augmented hierarchy*

In [20, Section 8], Masur–Minsky introduce the notion of *hierarchies without annuli*, which consist of tight geodesics on nonannular domains satisfying the usual subordinancy relations, where markings are replaced by pants decomposition. Hierarchies without annuli are useful for studying the geometry of the pants graph $\mathcal{P}(S)$ and, via work of Brock [7], the Weil–Petersson metric on $\mathcal{T}(S)$. Every hierarchy determines a unique hierarchy without annuli and, as noted in [20, Section 8], the hierarchy machinery translates seamlessly to the nonannular setting. The key idea is that nearly every relevant piece of information encoded in a hierarchy is determined by its nonannular geodesics, with the annular geodesics playing a peripheral role.

In this subsection, we explain how to associate a hierarchy to any augmented hierarchy. Unlike with hierarchies without annuli, this process will not be unique. Nonetheless, it will provide us a framework upon which to rebuild the work from [20, Sections 4 and 5] in our setting.

Let $\widetilde{H}$ be an augmented hierarchy between $\widetilde{\mu}, \widetilde{\eta} \in \mathcal{AM}(S)$. For each nonannular geodesic $\tilde{g}_Y \in \widetilde{H}$, relabel it as g_Y, and assign it new initial and terminal markings by $\mathbf{I}(g_Y) = \pi_{\mathcal{M}(Y)}(\widetilde{\mathbf{I}}(\tilde{g}_Y))$ and $\mathbf{T}(g_Y) = \pi_{\mathcal{M}(Y)}(\widetilde{\mathbf{T}}(\tilde{g}_Y))$, respectively. Let H_0 be the collection of the nonannular $g_Y \in \widetilde{H}$ with these new initial and terminal markings; these geodesics are tight in the original sense of [20, Definition 4.2]. The following lemma confirms that H_0 is a partial hierarchy.

Lemma 4.4. *H_0 is a partial hierarchy.*

Proof. We must prove that H_0 satisfies properties (1), (3), and the uniqueness part of (2) of [20, Definition 4.4]. Property (1) is obvious from the definition.

To see (3), suppose that $g'_Y \in H_0$. Then there is a $g_Y \in \widetilde{H}$ with $D(g_Y) = D(g'_Y)$. Since $\widetilde{H}$ is an augmented hierarchy, there exist $g_X, g_Z \in \widetilde{H}$ with $g_X \swarrow g_Y \searrow g_Z$. In particular, $\widetilde{\mathbf{I}}(g_Y) = \widetilde{\mathbf{I}}(Y, g_X)$ and $\widetilde{\mathbf{T}}(g_Y) = \widetilde{\mathbf{T}}(Y, g_Z)$. By definition, $\mathbf{I}(g'_Y) = \pi_{\mathcal{M}(Y)}(\widetilde{\mathbf{I}}(g'_Y)) = \pi_{\mathcal{M}(Y)}(\widetilde{\mathbf{I}}(Y, g_X)) = \mathbf{I}(Y, g_X)$, which is nonempty if and only if $\widetilde{\mathbf{I}}(Y, g_X)$ is. Thus $g'_X \swarrow g'_Y$ and similarly $g'_Y \searrow g'_Z$.

A similar argument shows that the uniqueness part of (2) holds. □

The unutilized configurations in H_0 are precisely the annular domains whose cores are curves appearing along geodesics in H_0, which coincide with those annular domains supporting geodesics in $\widetilde{H}$. For each unutilized configuration (Y, g_X, g_Z) in H_0, where Y is an annulus with core α, let $\tilde{g}_Y \in \widetilde{H}$ be the geodesic in $\mathcal{H}(\alpha)$, with initial and terminal vertices $\tilde{g}_{Y,\mathrm{int}}, \tilde{g}_{Y,\mathrm{ter}} \in \tilde{g}_Y$. Choose a tight geodesic g_Y between $\pi_{\mathcal{C}(\alpha)}(\tilde{g}_{Y,\mathrm{int}})$ and $\pi_{\mathcal{C}(\alpha)}(\tilde{g}_{Y,\mathrm{ter}})$, with $\mathbf{I}(g_Y) = \mathbf{I}(Y, g_X)$ and $\mathbf{T}(g_Y) = \mathbf{T}(Y, g_X)$. It follows from the proof of [**20**, Theorem 4.6] that the result from adding these tight geodesics to H_0 is a hierarchy, H. We call H a *hierarchy associated to* $\widetilde{H}$.

The following proposition describes the relationship between an augmented hierarchy and any hierarchy associated to it.

PROPOSITION 4.5. *Let $\widetilde{H}$ be an augmented hierarchy between $\widetilde{\mu}, \widetilde{\eta} \in \mathcal{AM}(S)$ and let H be any hierarchy associated to $\widetilde{H}$. Then the following hold.*

(i) *The map $\Phi : \widetilde{H} \to H$ given by $\Phi(\tilde{g}_Y) = g_Y$ is a bijection.*

(ii) *For any $\tilde{g}_Y \in \widetilde{H}$, we have $g_{Y,\mathrm{int}} = \pi_{\mathcal{C}(Y)}(\tilde{g}_{Y,\mathrm{int}})$ and $g_{Y,\mathrm{ter}} = \pi_{\mathcal{C}(Y)}(\tilde{g}_{Y,\mathrm{ter}})$, where $\tilde{g}_{Y,\mathrm{int}}, \tilde{g}_{Y,\mathrm{ter}} \in \tilde{g}_Y$ are its initial and terminal vertices.*

(iii) *For any $\tilde{g}_Y \in \widetilde{H}$, we have $\mathbf{I}(g_Y) = \pi_{\mathcal{M}(Y)}(\widetilde{\mathbf{I}}(\tilde{g}_Y))$ and $\mathbf{T}(g_Y) = \pi_{\mathcal{M}(Y)}(\widetilde{\mathbf{T}}(\tilde{g}_Y))$.*

(iv) *For any triple $\tilde{g}_X, \tilde{g}_Y, \tilde{g}_Z \in \widetilde{H}$, we have $\tilde{g}_X \swarrow \tilde{g}_Y \searrow \tilde{g}_Z$ in $\widetilde{H}$ if and only if $g_X \swarrow g_Y \searrow g_Z$ in H.*

Proof. (1) and (3) follow from the definition. To see (2), simply observe that $\tilde{g}_{Y,\mathrm{int}} = \pi_{\mathcal{C}(Y)}(\tilde{g}_{Y,\mathrm{int}})$ and $\tilde{g}_{Y,\mathrm{ter}} = \pi_{\mathcal{C}(Y)}(\tilde{g}_{Y,\mathrm{ter}})$ when Y is nonannular, and the relation holds by construction when Y is an annulus. To see (4), observe that $\mathbf{I}(g_Y) = \mathbf{I}(Y, g_X) = \pi_{\mathcal{M}(Y)}(\widetilde{\mathbf{I}}(Y, \tilde{g}_X)) = \pi_{\mathcal{M}(Y)}(\widetilde{\mathbf{I}}(\tilde{g}_Y))$ and $\mathbf{T}(g_Y) = \mathbf{T}(Y, g_Z) = \pi_{\mathcal{M}(Y)}(\widetilde{\mathbf{T}}(Y, \tilde{g}_Z)) = \pi_{\mathcal{M}(Y)}(\widetilde{\mathbf{T}}(\tilde{g}_Y))$. Since $\pi_{\mathcal{M}(Y)}(\widetilde{\mathbf{I}}(Y, \tilde{g}_X)) \neq \emptyset$ and $\pi_{\mathcal{M}(Y)}(\widetilde{\mathbf{T}}(Y, \tilde{g}_Z)) \neq \emptyset$ if and only if $\widetilde{\mathbf{I}}(Y, \tilde{g}_X) \neq \emptyset$ and $\widetilde{\mathbf{T}}(Y, \tilde{g}_Z) \neq \emptyset$, (4) follows. $\square$

Note that the above correspondence of subordinancy is independent of how we complete H_0 to a hierarchy H. Indeed, all the relevant data are contained in H_0.

4.5. *Augmenting the hierarchical technicalities*

In this subsection, we sketch the translation of [**20**, Section 4] to the augmented setting. As with hierarchies without annuli, most of the main constructions adapt without alteration. As such, the content of this subsection is mostly a series of observations and applications of Proposition 4.5.

We begin with an augmented version of [**20**, Theorem 4.7]. Given a domain $Y \subset S$ and an augmented hierarchy $\widetilde{H}$, let

$$\widetilde{\Sigma}^-(Y) = \{\tilde{g}_Z \in \widetilde{H} \,|\, Y \subset D(\tilde{g}_Z) \text{ and } \widetilde{\mathbf{I}}(\tilde{g}_Z)|_Y \neq \emptyset\}$$

and

$$\widetilde{\Sigma}^+(Y) = \{\tilde{g}_X \in \widetilde{H} \,|\, Y \subset D(\tilde{g}_X) \text{ and } \widetilde{\mathbf{T}}(\tilde{g}_X)|_Y \neq \emptyset\}.$$

These are the *forward* and *backward sequences* of Y, respectively. The following is the augmented analogue of [**20**, Theorem 4.7].

THEOREM 4.6 (Structure of sigma). *Let $\widetilde{H}$ be an augmented hierarchy and Y be any subsurface.*

(i) *If $\widetilde{\Sigma}^-(Y)$ is nonempty, then it has the form of a sequence: $\tilde{g}_{\widetilde{H}} = \tilde{g}_{X_n} \swarrow \cdots \swarrow \tilde{g}_{X_0}$. Similarly, if $\widetilde{\Sigma}^+(Y)$ is nonempty, then it has the form of a sequence: $\tilde{g}_{Z_0} \searrow \cdots \searrow \tilde{g}_{Z_m} = \tilde{g}_{\widetilde{H}}$.*

(ii) *If* $\widetilde{\Sigma}^{\pm}(Y)$ *are both nonempty, and* $\xi(Y) \neq 3$, *then* $\tilde{g}_{X_0} = \tilde{g}_{Z_0}$, *and* Y *intersects every vertex of* $\tilde{g}_{X_0}$ *nontrivially.*

(iii) *If* Y *is a component domain of any geodesic* $\tilde{g}_W \in \widetilde{H}$ *and* $\xi(Y) \neq 3$, *then*

$$\tilde{g}_X \in \widetilde{\Sigma}^-(Y) \Leftrightarrow \tilde{g}_X \swarrow \cdots \swarrow Y \quad \text{and} \quad \tilde{g}_Z \in \widetilde{\Sigma}^+(Y) \Leftrightarrow Y \searrow \cdots \searrow \tilde{g}_Z.$$

If, furthermore, $\widetilde{\Sigma}^{\pm}(Y)$ *are both nonempty, then* $X_0 = Y = Z_0$.

(iv) *Geodesics in* $\widetilde{H}$ *are determined by their support; that is, if* $\tilde{g}_X, \tilde{g}_Z \in \widetilde{H}$ *have* $X = Z$, *then* $\tilde{g}_X = \tilde{g}_Z$.

Proof. Let H be a hierarchy associated to $\widetilde{H}$ as constructed in Subsection 4.4. The proof is an easy application of [**20**, Theorem 4.7] to H and Proposition 4.5. ☐

We say that an augmented hierarchy $\widetilde{H}$ is *complete* if, for every subsurface Y with $\xi(Y) \neq 3$, if Y is a component domain of some geodesic in $\widetilde{H}$, then Y is the support of some geodesic in $\widetilde{H}$. The following is an immediate consequence of Theorem 4.6.

LEMMA 4.7. *Given any augmented hierarchy, if* $\widetilde{\mathbf{I}}(\widetilde{H})$ *and* $\widetilde{\mathbf{T}}(\widetilde{H})$ *are complete augmented markings, then* $\widetilde{H}$ *is complete.*

Proof. If $\xi(Y) \neq 3$, then both $\widetilde{\mathbf{I}}(\widetilde{H})|_Y, \widetilde{\mathbf{T}}(\widetilde{H})|_Y \neq \emptyset$. Thus $\tilde{g}_{\widetilde{H}} \in \widetilde{\Sigma}^+(Y), \widetilde{\Sigma}^-(Y)$, and so Y supports a geodesic in $\widetilde{H}$ by Theorem 4.6(2). ☐

We now construct augmented versions of the tools that originally went into proving [**20**, Theorem 4.7], as we need them in the next section. For the rest of the subsection, fix a hierarchy H associated to $\widetilde{H}$.

Recall the definition of a *footprint* of a subsurface on a geodesic. For any subsurface $Y \subset S$ and geodesic $\tilde{g}_X \in \widetilde{H}$ with X nonannular, let $\phi_{\tilde{g}_X}(Y)$ be the set of vertices of $\tilde{g}_X$ disjoint from Y; if Y is an annulus with core α, then $\phi_{\tilde{g}_X}(Y)$ are simply those vertices of $\tilde{g}_X$ disjoint from α. If $g_X \in H$ is the geodesic corresponding to $\tilde{g}_X$, then $\phi_{\tilde{g}_X}(Y) = \phi_{g_X}(Y)$. We note that augmented versions of [**20**, Lemma 4.10 and Corollary 4.11] follow immediately from this observation.

Masur–Minsky define two partial orders on geodesics in a hierarchy which we will recall and redefine for augmented hierarchies. We will show that the correspondence between H and $\widetilde{H}$ preserves these orders. The first is time order [**20**, Definition 4.16].

DEFINITION 4.8 (Time order). Given two geodesics $\tilde{g}_X, \tilde{g}_Z \in \widetilde{H}$, we say $\tilde{g}_X$ is *time-ordered* before $\tilde{g}_Z$ and write $\tilde{g}_X \prec_t \tilde{g}_Z$ if there is a geodesic $\tilde{g}_Y \in \widetilde{H}$ with $X, Z \subset Y$ and $\max \phi_{\tilde{g}_Z}(X) < \min \phi_{\tilde{g}_Z}(Y)$.

Observe that if $\tilde{g}_X \prec_t \tilde{g}_Z$ and $g_X, g_Z, g_Y \in H$ are the corresponding geodesics, then $\max \phi_{g_Z}(X) = \max \phi_{\tilde{g}_Z}(X) < \min \phi_{\tilde{g}_Z}(Y) = \min \phi_{g_Z}(Y)$, and so $\tilde{g}_X \prec_t \tilde{g}_Z$ if and only if $g_X \prec_t g_Z$.

Given a geodesic $\tilde{g}_Y \in \widetilde{H}$, a *position* on $\tilde{g}_Y$ is either a vertex or one of $\widetilde{\mathbf{I}}(\tilde{g}_Y)$ or $\widetilde{\mathbf{T}}(\tilde{g}_Y)$. We can extend the natural linear order on the vertices $\tilde{g}_Y$ to a linear order on positions by taking $\widetilde{\mathbf{I}}(\tilde{g}_Y) < v < \widetilde{\mathbf{T}}(\tilde{g}_Y)$ for all $v \in \tilde{g}_Y$. A *pointed geodesic* is a pair $(\tilde{g}_Y, v)$, where v is some position on $\tilde{g}_Y$.

We can define a notion of footprint on pointed geodesics as follows: Given a pointed geodesic $(\tilde{g}_Y, v)$ and a geodesic $\tilde{g}_X \in \widetilde{H}$, we set

$$\hat{\phi}_{\tilde{g}_X}(\tilde{g}_Y, v) = \begin{cases} \phi_{\tilde{g}_X}(Y) & \text{if } Y \subset X, \\ v & \text{if } X = Y. \end{cases}$$

If $g_X, g_Y \in H$ are the geodesics corresponding to $\tilde{g}_X, \tilde{g}_Y \in \widetilde{H}$, then it is clear that $\hat{\phi}_{\tilde{g}_X}(\tilde{g}_Y, v) = \hat{\phi}_{g_X}(g_Y, v)$ unless $X = Y$ is an annulus, in which case $\prec_p$ restricts to the linear orders on positions of $\tilde{g}_X$ and g_X.

We can now define a partial order on pointed geodesics.

Definition 4.9. Given two pointed geodesics $(\tilde{g}_X, v_X), (\tilde{g}_Z, v_Z)$, we write $(\tilde{g}_X, v_X) \prec_p (\tilde{g}_Z, v_Z)$ if and only if there exists some geodesic $\tilde{g}_Y \in \widetilde{H}$ with $\tilde{g}_X \underset{=}{\searrow} \cdots \underset{=}{\searrow} \tilde{g}_Y \underset{=}{\swarrow} \cdots \underset{=}{\swarrow} \tilde{g}_Z$ and

$$\max \hat{\phi}_{\tilde{g}_Y}(\tilde{g}_X, v_X) < \min \hat{\phi}_{\tilde{g}_Y}(\tilde{g}_Z, v_Z).$$

If $g_X, g_Y, g_Z \in H$ are the geodesics corresponding to $\tilde{g}_X, \tilde{g}_Y, \tilde{g}_Z \in \widetilde{H}$, then observe that $g_X \underset{=}{\searrow} \cdots \underset{=}{\searrow} g_Y \underset{=}{\swarrow} \cdots \underset{=}{\swarrow} g_Z$ and $\max \hat{\phi}_{g_Y}(g_X, v_X) = \max \hat{\phi}_{\tilde{g}_Y}(\tilde{g}_X, v_X) < \min \hat{\phi}_{\tilde{g}_Y}(\tilde{g}_Z, v_Z) = \min \hat{\phi}_{g_Y}(g_Z, v_Z)$, so that $(\tilde{g}_X, v_X) \prec_p (\tilde{g}_Z, v_Z)$ if and only if $(g_X, v_X) \prec_p (g_Z, v_Z)$, unless $X = Y = Z$ is an annulus, in which case $\prec_p$ is again just the linear orders on positions of $\tilde{g}_X$ and g_X.

We have shown the following lemma.

Lemma 4.10. *Let $\widetilde{H}$ be an augmented hierarchy and H be any associated hierarchy. Then the following conditions are satisfied.*

(i) *Both $\prec_t$ and $\prec_p$ are strict partial orders.*

(ii) *For any $\tilde{g}_X, \tilde{g}_Y \in \widetilde{H}$ with corresponding geodesics $g_X, g_Y \in H$, then*

$$\tilde{g}_X \prec_t \tilde{g}_Y \Leftrightarrow g_X \prec_t g_Y.$$

(iii) *If in addition X and Y are nonannular, then*

$$(\tilde{g}_X, x) \prec_p (\tilde{g}_Y, y) \Leftrightarrow (g_X, x) \prec_p (g_Y, y).$$

As with hierarchies, we have the following four mutually exclusive cases for $(\tilde{g}_X, x) \prec_p (\tilde{g}_Y, y)$:

(1) $\tilde{g}_X \prec_t \tilde{g}_Y$;
(2) $\tilde{g}_X = \tilde{g}_Y$ and $x < y$;
(3) $\tilde{g}_X \searrow \cdots \searrow \tilde{g}_Y$ and $\max \phi_{\tilde{g}_Y}(X) < y$;
(4) $\tilde{g}_X \swarrow \cdots \swarrow \tilde{g}_Y$ and $x < \min \phi_{\tilde{g}_X}(Y)$.

We think of a pointed geodesic as giving a position on a geodesic in $\widetilde{H}$, so that $\prec_p$ gives a partial order on positions on a geodesic. In the next section, we describe how to build coordinates, called slices, on an augmented hierarchy, which are special arrangements of these positions. We will upgrade $\prec_p$ to a partial order on these coordinates, which we can then use to build paths in $\mathcal{AM}(S)$ which make definite progress through the augmented hierarchy.

5. *Augmented hierarchy paths*

In this section, we explain how to build augmented hierarchy paths from augmented hierarchies. Similar to hierarchy paths, this process involves resolving an augmented hierarchy into a sequence of slices, then finding a sequence of associated augmented markings which we connect with boundedly many elementary moves in $\mathcal{AM}(S)$.

5.1. *Augmented slices*

In this subsection, we develop the notion of a slice of an augmented hierarchy, which is roughly a way of giving coordinates in the augmented hierarchy which respect the subordinancy relations. The definition of a slice of a hierarchy [**20**, Section 5] is the same as that of an augmented slice, except that one takes geodesics in combinatorial horoballs over annular curve graphs instead.

DEFINITION 5.1 (Augmented slices). An *augmented slice* $\widetilde{\tau}$ of an augmented hierarchy $\widetilde{H}$ is a collection of pairs $(\tilde{g}_X, x)$ with $x \in \tilde{g}_X \in \widetilde{H}$ satisfying the following.

(S1) A geodesic $\tilde{g}_X$ appears at most once in $\widetilde{\tau}$.
(S2) There is a distinguished pair $(\tilde{g}_{\widetilde{\tau}}, v_{\widetilde{\tau}}) \in \widetilde{\tau}$ called the *bottom pair* of $\widetilde{\tau}$ and $\tilde{g}_{\widetilde{\tau}}$ is the *bottom geodesic*.
(S3) For every pair $(\tilde{g}_Y, y) \in \widetilde{\tau}$ other than the bottom pair, there is a pair $(\tilde{g}_X, x) \in \widetilde{\tau}$ of which Y is a component domain.

We say that $\widetilde{\tau}$ is *complete* if

(S4) Given a pair $(\tilde{g}_Y, y) \in \widetilde{\tau}$, for every component domain X of $(\tilde{g}_Y, v)$, there exists a pair $(\tilde{g}_X, x) \in \widetilde{\tau}$.

An augmented slice $\widetilde{\tau}$ is called *initial* if, for each pair we have $(\tilde{g}_Y, y) \in \widetilde{\tau},\ y = \tilde{g}_{Y,\mathrm{int}}$. A complete initial slice is uniquely determined by its bottom geodesic, and $\widetilde{H}$ has a unique initial slice with bottom geodesic $\tilde{g}_{\widetilde{H}}$. We can define *terminal* augmented slices similarly.

To each augmented slice $\widetilde{\tau}$, there is a unique way to associate an augmented marking $\widetilde{\mu}_{\widetilde{\tau}}$ as follows: First, observe by induction that the vertices α appearing in nonannular geodesics in $\widetilde{\tau}$ are disjoint and distinct, so that they form a maximal simplex in $\mathcal{C}(S)$, which we make $\{\mathrm{base}(\widetilde{\mu}_{\widetilde{\tau}})$. We can then associate transversal and length coordinates to each base curve $\alpha \in \{\mathrm{base}(\widetilde{\mu}_{\widetilde{\tau}})$ if $\widetilde{\tau}$ contains a pair $(\tilde{g}_X, x)$ with $x = (t_\alpha, D_\alpha)$, where X is an annulus with core α, by choosing t_α and D_α as the transversal and length coordinate for α in $\widetilde{\mu}_{\widetilde{\tau}}$. Note that a complete slice determines a complete augmented marking. Typically, this underlying marking is not clean, so one can clean the transversals to base curves by choosing new transversals that minimize the distance in the corresponding annular curve graphs. We say that any such complete, clean augmented marking is *compatible* with its associated slice. The number of such compatible augmented markings is uniformly bounded, similar to [**20**, Lemma 2.4].

LEMMA 5.2. *There exists $C' > 0$, depending only on S such that, for any augmented slice $\widetilde{\tau}$ of an augmented hierarchy $\widetilde{H}$, the number of augmented markings compatible with $\widetilde{\tau}$ is less than C, each of which differs by a bounded number of twist moves.*

Proof. Fix a clean augmented marking $\widetilde{\mu}$ compatible with $\widetilde{\tau}$. Then $\mathrm{base}(\widetilde{\mu}) = \mathrm{base}(\widetilde{\mu}_{\widetilde{\tau}})$ and $D_\alpha(\widetilde{\mu}) = D_\alpha(\widetilde{\mu}_{\widetilde{\tau}})$ for all $\alpha \in \mathcal{C}(S)$ by definition. Because $\mathcal{C}(\alpha) \asymp \mathbb{Z}$, for each triple $(\alpha, t_\alpha, D_\alpha) \in \widetilde{\mu}_{\widetilde{\tau}}$, there is a choice of clean transversal $\beta \in \mathcal{C}(\alpha)$, which minimizes $d_\alpha(t_\alpha, \pi_\alpha(\beta)$, where the minimum is uniformly bounded, completing the proof. $\square$

5.2. *Partial order on slices*

In [**20**, Section 5], Masur–Minsky define a partial order on the set of complete slices of H. We now do this for augmented slices.

Let $\widetilde{V}(\widetilde{H})$ be the set of complete augmented slices on $\widetilde{H}$. Given $\widetilde{\tau}, \widetilde{\tau}' \in \widetilde{V}(\widetilde{H})$, we say $\widetilde{\tau} \prec_s \widetilde{\tau}'$ if and only if $\widetilde{\tau} \neq \widetilde{\tau}'$ and, for any $(\tilde{g}_Y, y) \in \widetilde{\tau}$, either $(\tilde{g}_Y, y) \in \widetilde{\tau}'$ or there is some $(\tilde{g}_X, x) \in \widetilde{\tau}'$ with $(\tilde{g}_Y, y) \prec_p (\tilde{g}_X, x)$.

Lemma 5.3. *Let $\widetilde{H}$ be an augmented hierarchy. Then $\prec_s$ is a strict partial order on $\widetilde{V}(\widetilde{H})$.*

Proof. We proceed as in the proof of [**20**, Lemma 5.1] by showing that $\prec_s$ is transitive, since it is never reflexive by definition. Suppose $\widetilde{\tau}_1 \prec_s \widetilde{\tau}_2 \prec_s \widetilde{\tau}_3$ for $\widetilde{\tau}_i \in \widetilde{V}(\widetilde{H})$.

By definition of $\prec_s$, for $i = 1, 2$, given any pair $p_i \in \widetilde{\tau}_i$, there exists a pair $p_{i+1} \in \widetilde{\tau}_{i+1}$ such that either $p_i \prec_p p_{i+1}$ or $p_i = p_{i+1}$. Since $\prec_p$ is a strict partial order (Lemma 4.10), either $p_1 \prec_p p_3$ or $p_1 = p_3$, implying either $\widetilde{\tau}_1 \prec_s \widetilde{\tau}_3$ or $\widetilde{\tau}_1 = \widetilde{\tau}_3$. Since augmented slices in $\widetilde{V}(\widetilde{H})$ are complete, we must have some $p_1 \in \widetilde{\tau}_1$ with $p_1 \notin \widetilde{\tau}_2$. Thus $p_1 \prec_p p_2$ and thus $p_1 \prec_p p_3$, implying $p_3 \notin \widetilde{\tau}_1$, since pairs in the same slice are not $\prec_p$-comparable by [**20**, Lemmas 4.18(1) and 4.19], which hold for augmented hierarchies by Lemma 4.10. $\square$

5.3. *Elementary moves of augmented slices*

In this section, we describe, following [**20**, Section 5], how to resolve an augmented hierarchy into a sequence of complete augmented slices which are related by certain elementary moves, which we define shortly. Informally, an elementary move of augmented slices is one which makes progress by one vertex along some geodesic in $\widetilde{H}$. First, we need to define transition slices, which will record the reorganization that accompanies this progress.

Let $\tilde{g}_X \in \widetilde{H}$ and suppose $x \in \tilde{g}_X$ is not the last vertex of $\tilde{g}_X$, with x' its successor. We presently define *transition slices* for x and x', $\widetilde{\sigma}$ and $\widetilde{\sigma}'$, which have the property that $\widetilde{\mu}_{\widetilde{\sigma}} = \widetilde{\mu}_{\widetilde{\sigma}'} = x \cup x'$ when $\xi(X) > 4$.

Let $\widetilde{\sigma}$ be the smallest slice with bottom pair $(\tilde{g}_X, x)$ such that, for any $(\tilde{g}_Z, z) \in \widetilde{\tau}$ and Y a component domain of (Z, z), the following properties hold:

(E1) If $x'|_Y \neq \emptyset$ and Y supports a geodesic $\tilde{g}_Y \in \widetilde{H}$, then $(\tilde{g}_Y, y) \in \widetilde{\sigma}$, where y is the *terminal* vertex of $\tilde{g}_Y$.
(E2) If $x'|_Y = \emptyset$, then no geodesic in Y is included in $\widetilde{\sigma}$.

One builds $\widetilde{\sigma}$ inductively and confirms easily that it satisfies (S1)–(S3) of Definition 5.1. We call the domains in (E2) *unused domains* for $\widetilde{\sigma}$. Similarly, we may define $\widetilde{\sigma}'$ as the smallest slice with bottom pair $(\tilde{g}_X, x')$, such that, for any $(\tilde{g}_Z, z) \in \widetilde{\tau}'$ and Y, a component domain of (Z, z), the following propositions hold.

(E1') If $x|_Y \neq \emptyset$ and Y supports a geodesic $\tilde{g}_Y \in \widetilde{H}$, then $(\tilde{g}_Y, y) \in \widetilde{\sigma}$, where y is the *initial* vertex of $\tilde{g}_Y$.
(E2') If $x|_Y = \emptyset$, then no geodesic in Y is included in $\widetilde{\sigma}$.

We remark on transition slices for $y, y' \in \tilde{g}_Y \in \widetilde{H}$ with $\xi(Y) \leqslant 4$.

(1) If Y is an annulus, then $\widetilde{\sigma} = \{(\tilde{g}_Y, y)\}$ and $\widetilde{\sigma}' = \{(\tilde{g}_Y, y')\}$.
(2) If Y is a once-punctured torus, then y and y' intersect in Y. Let X and X' be annuli with cores y, y', respectively. Then $\widetilde{\sigma} = \{(\tilde{g}_Y, y), (\tilde{g}_X, \pi_{\mathcal{H}_y}(y'))\}$ and $\widetilde{\sigma}' = \{(\tilde{g}_Y, y'), (\tilde{g}_{X'}, \pi_{\mathcal{H}_{y'}}(y))\}$.
(3) If Y is a four-holed sphere, then y and y' intersect twice, so $\pi_X(y') = \widetilde{\mathbf{T}}(\tilde{g}_X)$ has two components, one of which is the last vertex of $\tilde{g}_X$.

The following lemma characterizes transition slices for most geodesics and is a restatement and direct consequence of [**20**, Lemma 5.2].

Lemma 5.4. *Let y, y' be successive vertices along a geodesic $\tilde{g}_Y \in \widetilde{H}$ with $\xi(Y) > 4$, and let $\widetilde{\sigma}, \widetilde{\sigma}'$ be the associated transition slices. Then no geodesics in $\widetilde{\sigma}$ and $\widetilde{\sigma}'$ have annular domains, the associated augmented markings $\widetilde{\mu}_{\widetilde{\sigma}}$ and $\widetilde{\mu}_{\widetilde{\sigma}'}$ have no transversals and are both equal to $y \cup y'$, and the unused domains in $\widetilde{\sigma}$ and $\widetilde{\sigma}'$ are exactly the component domains of $(Y, y \cup y')$.*

Proof. Let H be any hierarchy associated to $\widetilde{H}$. Let $\tilde{g}_Y \in \widetilde{H}$ with $\xi(Y) > 4$ and let $y \in \tilde{g}_Y$ be not the terminal vertex of $\tilde{g}_Y$ with successor $y' \in \tilde{g}_Y$. If $\widetilde{\sigma}, \widetilde{\sigma}'$ are the associated transition slices, set $\sigma = \{(g_Z, \pi_{\mathcal{C}(Z)}(z)) \,|\, (\tilde{g}_Z, z) \in \widetilde{\sigma}\}$ and $\sigma' = \{(g_Z, \pi_{\mathcal{C}(Z)}(z)) \,|\, (\tilde{g}_Z, z) \in \widetilde{\sigma}'\}$. It follows easily from Proposition 4.5 that σ and σ' are the transition slices for y, y' along g_Y. Thus the lemma follows from [**20**, Lemma 5.2]. $\square$

Definition 5.5 (Forward elementary move of augmented slices). Let y, y' be successive vertices along $\tilde{g}_Y \in \widetilde{H}$ with transition slices $\widetilde{\sigma}, \widetilde{\sigma}'$. We say that two complete augmented slices $\widetilde{\tau}$ and $\widetilde{\tau}'$ are related by a *forward elementary move of augmented slices* along $\tilde{g}_Y$ from y to y' if $\widetilde{\sigma} \subset \widetilde{\tau}$, $\widetilde{\sigma}' \subset \widetilde{\tau}'$, and $\widetilde{\tau} \setminus \widetilde{\sigma} = \widetilde{\tau}' \setminus \widetilde{\sigma}'$.

The next lemma confirms that a forward elementary move in $\widetilde{V}(\widetilde{H})$ makes progress in $\prec_s$, as in [**20**, Lemma 5.3], whose proof is identical.

Lemma 5.6. *Suppose $\widetilde{\tau}, \widetilde{\tau}' \in \widetilde{V}(\widetilde{H})$ and are related by an elementary move $\widetilde{\tau} \to \widetilde{\tau}'$ along $\tilde{g}_Y \in \widetilde{H}$. Then $\widetilde{\tau} \prec_s \widetilde{\tau}'$.*

Proof. Since $\widetilde{\sigma} \neq \widetilde{\sigma}'$, we have $\widetilde{\tau} \neq \widetilde{\tau}'$. Let $(\tilde{g}_X, X) \in \widetilde{\tau}$ such that $(\tilde{g}_X, x) \notin \widetilde{\tau}'$. Then $(\tilde{g}_X, x) \in \widetilde{\sigma}$, and thus $X \subset Y$ and $y'|_X \neq \emptyset$, by definition of $\widetilde{\sigma}$. If $\tilde{g}_X = \tilde{g}_Y$, then $(\tilde{g}_X, x) = (\tilde{g}_Y, y) \prec_p (\tilde{g}_Y, y')$, and we are done. If not, then $\phi_{\tilde{g}_Y}(X)$ contains y and not y', so that $\max \phi_{\tilde{g}_Y}(X) = v < v'$, implying $(\tilde{g}_X, x) \prec_p (\tilde{g}_Y, y')$, completing the proof. $\square$

5.4. *Resolutions of augmented slices*

In this subsection, we prove that every complete augmented hierarchy $\widetilde{H}$ admits a sequence of elementary moves between its initial and terminal augmented slices, called a *resolution of* $\widetilde{H}$. Importantly, the length of any such resolution is bounded by $|\widetilde{H}| = \sum_{\tilde{g}_Y \in \widetilde{H}} |\tilde{g}_Y|$. The proof is a straightforward adaptation of [**20**, Proposition 5.4], so we leave some details to the reader.

Proposition 5.7 (Resolutions exist). *Any complete augmented hierarchy admits a sequence of forward elementary moves $\widetilde{\tau}_0 \to \cdots \to \widetilde{\tau}_N$, where $\widetilde{\tau}_0$ is the initial slice, $\widetilde{\tau}_N$ the terminal slice, and $N \leqslant |\widetilde{H}|$.*

Proof. First, suppose that $\widetilde{\tau} \in \widetilde{V}(\widetilde{H})$ is not the terminal slice of $\widetilde{H}$. Then there exists $(\tilde{g}_Y, y) \in \widetilde{\tau}$ such that y is not the terminal vertex of $\tilde{g}_Y$ with successor y'. Choose $\tilde{g}_Y$ minimally so that if $(\tilde{g}_X, x) \in \widetilde{\tau}$ and $X \subset Y$, then x is the terminal vertex of $\tilde{g}_X$. Because $\tilde{g}_Y$ is minimal and $\widetilde{\tau}$ is complete, the subset

$$\widetilde{\sigma} = \{(\tilde{g}_X, x) \in \widetilde{\tau} \,|\, X \subset Y, y'|_X \neq \emptyset\}$$

satisfies the two transition slice properties (E1) and (E2). Using (E1') and (E2'), one can build the other transition slice $\widetilde{\sigma}'$ for y and y'. Set $\widetilde{\tau}' = \widetilde{\sigma}' \cup (\widetilde{\tau} \setminus \widetilde{\sigma})$. One can confirm, as done in [**20**, Proposition 5.4], that $\widetilde{\tau}'$ is a complete augmented slice, thus making $\widetilde{\tau} \to \widetilde{\tau}'$ a forward elementary move.

This builds a sequence of slice $\widetilde{\tau}_0 \to \widetilde{\tau}_1 \to \cdots$, which terminates, say at $\widetilde{\tau}_N$, because each move makes progress with respect to $\prec_s$ and $\widetilde{V}(\widetilde{H})$ is finite. It remains to prove that $N \leqslant |\widetilde{H}|$.

To see this, suppose that $(\tilde{g}_Z, z) \in \widetilde{\tau}_n$ and $(\tilde{g}_Z, z') \in \widetilde{\tau}_m$ for $n < m$. Then $\widetilde{\tau}_n \prec_s \widetilde{\tau}_m$ and so $z \leqslant z'$. If not, then $(\tilde{g}_Z, z') \prec_p (\tilde{g}_Z, z)$, implying by definition of $\prec_s$ that there is some $(\tilde{g}_W, w) \in \widetilde{\tau}_m$ with $(\tilde{g}_Z, z) \prec_p (\tilde{g}_W, w)$, which is a contradiction of the fact that pairs in the same slice are not $\prec_p$-comparable, as in Lemma 5.3. This shows that vertices cannot reappear once traversed by the resolution process.

By definition, a forward elementary move advances exactly one step along a geodesic and replaces pairs $(\tilde{g}_Y, \tilde{g}_{Y,\mathrm{ter}})$ with pairs $(\tilde{g}_X, \tilde{g}_{X,int})$, leaving all other pairs fixed. It follows from the previous paragraph that $N \leqslant \sum_{\tilde{g}_Y \in \widetilde{H}} = |\widetilde{H}|$, completing the proof. $\square$

5.5. *Augmented hierarchy paths defined*

Given any augmented hierarchy $\widetilde{H}$, Proposition 5.7 builds a sequence $\widetilde{\tau}_0 \to \widetilde{\tau}_i \to \cdots \widetilde{\tau}_N$ of forward elementary moves, where $\widetilde{\tau}_0$ and $\widetilde{\tau}_N$ are the initial and terminal augmented slices of $\widetilde{H}$, respectively. For each i, let $\widetilde{\mu}_i$ be any augmented marking compatible with $\widetilde{\tau}_i$, choosing $\widetilde{\mu}_0 = \widetilde{\mu}$ and $\widetilde{\mu}_N = \widetilde{\eta}$. This gives a sequence of augmented markings $\widetilde{\mu} = \widetilde{\mu}_0 \to \cdots \to \widetilde{\mu}_N = \widetilde{\eta}$, which we call an *augmented hierarchy path* between $\widetilde{\mu}$ and $\widetilde{\eta}$.

Eventually, we will prove that augmented hierarchy paths are uniform quasigeodesics in $\mathcal{AM}(S)$. The following lemma, similar to [**20**, Lemma 5.5], is the first step in this process. It proves that each step in an augmented hierarchy path moves a uniformly bounded distance in $\mathcal{AM}(S)$.

LEMMA 5.8. *There exists a $B > 0$, depending only on S so that $d_{\mathcal{AM}(S)}(\widetilde{\mu}_i, \widetilde{\mu}_{i+1}) < B$ for each $i = 0, \ldots, N-1$.*

Proof. Suppose that $\widetilde{\tau}_i \to \widetilde{\tau}_{i+1}$ comes from a transition $y \to y'$ along $\tilde{g}_Y \in \widetilde{H}$. If Y is an annulus, let $y = (t_\alpha, D_\alpha(\widetilde{\mu}_i)) \in \mathcal{H}(\alpha)$ and $y' = (t'_\alpha, D_\alpha(\widetilde{\mu}_{i+1})) \in \mathcal{H}(\alpha)$. If $D_\alpha(\widetilde{\mu}) = D_\alpha(\widetilde{\mu}_{i+1})$, then $d_\alpha(t_\alpha, t'_\alpha) \leqslant 2^{D_\alpha(\widetilde{\mu}_i)}$, so a bounded number of twist moves in $\mathcal{AM}(S)$ yields an augmented marking $\widetilde{\mu}'_{i+1}$ compatible with $\widetilde{\tau}_{i+1}$. If $D_\alpha(\widetilde{\mu}_i) \neq D_\alpha(\widetilde{\mu}_{i+1})$, then $\widetilde{\tau}_i \to \widetilde{\tau}_{i+1}$ encodes a vertical move and $\pi_{\mathcal{M}(S)}(\widetilde{\mu}_i) = \pi_{\mathcal{M}(S)}(\widetilde{\mu}_{i+1})$, implying $d_{\mathcal{AM}(S)}(\widetilde{\mu}_i, \widetilde{\mu}_{i+1}) = 1$.

Now suppose that $\xi(Y) = 4$. Then recall from before that the transition slices are $\widetilde{\sigma}_i = \{(\tilde{g}_Y, y), (\tilde{g}_X, x)\}$ and $\widetilde{\sigma}_{i+1} = \{(\tilde{g}_Y, y'), (\tilde{g}_{X'}, x')\}$, where X and X' are annuli with cores y and y', respectively, and x and x' are vertices of $\pi_X(y')$ and $\pi_{X'}(y)$, respectively. Construct a clean augmented marking $\widetilde{\mu}'_i$ compatible with $\widetilde{\tau}_i$ which contains the triple $(y, \pi_X(y'), D_X(\widetilde{\mu}'_i))$, where $D_X(\widetilde{\mu}'_i) = 0$ necessarily. A flip move on $\widetilde{\mu}'_i$ along y results in an augmented marking $\widetilde{\mu}'_{i+1}$ with the triple $(y', \pi_{X'}(y), D_{X'}(\widetilde{\mu}'_{i+1}))$, with all other base curves of $\widetilde{\mu}'_{i+1}$ being the same as those of $\widetilde{\mu}'_i$, $D_\alpha(\widetilde{\mu}'_i) = D_\alpha(\widetilde{\mu}'_{i+1})$ for all $\alpha \in \mathcal{C}(S)$, and the transversals at uniformly bounded distance by Lemma 5.2. Thus $\widetilde{\mu}'_{i+1}$ is a uniformly bounded number of twist moves along the base curves from an augmented marking $\widetilde{\mu}''_{i+1}$ compatible with $\widetilde{\tau}_{i+1}$. Since the distance between augmented markings compatible with the same augmented slice is uniformly bounded by Lemma 5.2, this implies $d_{\mathcal{AM}(S)}(\widetilde{\mu}_i, \widetilde{\mu}_{i+1})$ is uniformly bounded.

Finally, if $\xi(Y) > 4$, then $\widetilde{\tau}_i$ and $\widetilde{\tau}_{i+1}$ have the same base curves and positions on their horoball geodesics. Thus $\widetilde{\mu}_i$ and $\widetilde{\mu}_{i+1}$ are both compatible with $\widetilde{\tau}_i$ and $\widetilde{\tau}_{i+1}$, implying that $d_{\mathcal{AM}(S)}(\widetilde{\mu}_i, \widetilde{\mu}_{i+1})$ is uniformly bounded in this case again by Lemma 5.2, completing the proof. $\square$

6. *Length and efficiency of augmented hierarchy paths*

In this section, we convert the structural results in the previous section to prove that augmented hierarchy paths are uniform quasigeodesics in $\mathcal{AM}(S)$, from which we give a combinatorial proof of Rafi's distance formula for $\mathcal{T}(S)$, Theorem 2.10.

6.1. *Projecting augmented markings to subsurfaces*

In this subsection, we define subsurface projections for augmented markings, the $\mathcal{AM}(S)$-analogue of those for markings, as in Definition 2.2.

Let $Y \subset S$ be any subsurface and $\widetilde{\mu} \in \mathcal{AM}(S)$ be any augmented marking. If Y is an annulus with core α, then set $\pi_Y(\widetilde{\mu}) = \pi_{\mathcal{H}(\alpha)}(\widetilde{\mu}) = (\pi_\alpha(\widetilde{\mu}), D_\alpha(\widetilde{\mu})) \in \mathcal{H}(\alpha)$. If Y is nonannular, set $\pi_Y(\widetilde{\mu}) = \pi_Y(\pi_{\mathcal{M}(S)}(\widetilde{\mu}))$.

The following lemma proves that subsurface projections are 4-lipschitz.

Lemma 6.1 ([20, Lemma 2.3]). *For any $\widetilde{\mu} \in \mathcal{AM}(S)$ and subsurface $Y \subset S$,* $\operatorname{diam}_Y(\pi_Y(\widetilde{\mu})) < 4$.

Proof. The only case left to consider is when Y is an annulus with core α. Then

$$\operatorname{diam}_{\mathcal{H}(\alpha)}(\pi_{\mathcal{H}(\alpha)}(\widetilde{\mu})) \leqslant \operatorname{diam}_\alpha(\pi_\alpha(\pi_{\mathcal{M}(S)}(\widetilde{\mu}))) \leqslant 4,$$

completing the proof. □

Similarly, if two augmented markings are connected via an elementary move, then their projection to any subsurface is uniformly bounded.

Lemma 6.2 ([20, Lemma 2.5]). *Suppose $\widetilde{\mu}_1, \widetilde{\mu}_2 \in \mathcal{AM}(S)$ are connected via an elementary move. Then $d_Y(\widetilde{\mu}_1, \widetilde{\mu}_2) < 4$ for any nonpants $Y \subset S$.*

Proof. The remaining case is when Y is an annulus about a curve α and $\mathcal{C}(Y) = \mathcal{H}(\alpha)$. If the move is a vertical or twist move about α, then $d_{\mathcal{H}(\alpha)}(\widetilde{\mu}_1, \widetilde{\mu}_2) = 1$. Otherwise, [20, Lemma 2.5] implies $d_\alpha(\widetilde{\mu}_1, \widetilde{\mu}_2) \leqslant 4$, from which it follows that $d_{\mathcal{H}(\alpha)}(\widetilde{\mu}_1, \widetilde{\mu}_2) \leqslant 4$ by definition of $\mathcal{H}(\alpha)$. □

Given two subsurfaces $X, Y \subset S$, we write $X \pitchfork Y$ if $X \cap Y \neq \emptyset$ and neither is contained in the other. The following lemma is due to Behrstock [1], but the effective bound is due to Leininger [17]. It holds for augmented markings by definition of the subsurface projection.

Lemma 6.3 (Behrstock's inequality). *If $X \pitchfork Y$ with $\xi(X), \xi(Y) \geqslant 4$, then, for any $\widetilde{\mu} \in \mathcal{AM}(S)$, we have*

$$\min\{d_Y(\widetilde{\mu}, \partial X), d_X(\widetilde{\mu}, \partial Y)\} < 10.$$

One of the key tools of [20] is the following theorem.

Theorem 6.4 (Bounded geodesic image theorem; [20, Theorem 3.1]). *There is a constant $M_0 > 0$ such that the following holds. Let $\gamma \subset \mathcal{C}(S)$ be any geodesic and $Y \subset S$ be any subsurface. If $d_{\mathcal{C}(S)}(\gamma, \partial Y) > 1$, then $\operatorname{diam}_{\mathcal{C}(Y)}(\gamma) < M_0$.*

Proof. We need only prove it when $Y = \mathcal{H}_\alpha$ for some $\alpha \in \mathcal{C}(S)$. Since $d_{\mathcal{C}(S)}(\gamma, \alpha) > 1$, it follows that $D_\alpha(\gamma_i) = 0$ for each $\gamma_i \in \gamma$, and so $\operatorname{diam}_{\mathcal{H}_\alpha}(\gamma) \asymp \log \operatorname{diam}_{\mathcal{C}(\alpha)}(\gamma) < \operatorname{diam}_{\mathcal{C}(\alpha)}(\gamma) \asymp 1$, completing the proof. □

6.2. *The forward and backward paths of a subsurface*

Let $Y \subset S$ be any subsurface. In this subsection, we will show how to convert $\widetilde{\Sigma}^+(Y)$ and $\widetilde{\Sigma}^-(Y)$ into sets of pointed geodesics, which package all the relevant combinatorial information in $\widetilde{H}$ about Y. In the next subsection, we will use these packages to prove a version of the Large Links Lemma 2.5 for $\mathcal{AM}(S)$ and augmented hierarchies.

We proceed as in [**20**, Subsection 6.1]. First, recall that Theorem 4.6 implies that $\widetilde{\Sigma}^+(Y)$ has the form $\tilde{g}_{Z_0} \searrow \cdots \tilde{g}_{Z_n} = \tilde{g}_{\widetilde{H}}$, and $\widetilde{\Sigma}^-(Y)$ has the form $\tilde{g}_{\widetilde{H}} = \tilde{g}_{X_m} \swarrow \cdots \swarrow \tilde{g}_{X_0}$. Let

$$\sigma = \{(\tilde{g}_Z, z) \,|\, z \in \tilde{g}_Z \in \widetilde{\Sigma}^{\pm}(Y) \text{ and } z|_Y \neq \emptyset\}.$$

LEMMA 6.5. *The partial order $\prec_p$ restricts to a linear order on σ.*

Proof. For each $\tilde{g}_{Z_i} \in \widetilde{\Sigma}^+(Y)$, let $z_i \in \tilde{g}_{Z_i}$ be the position immediately following $\max \phi_{\tilde{g}_{Z_i}}(Y)$ (or $z_i = \widetilde{\mathbf{T}}(\tilde{g}_{Z_i})$ if $\max \phi_{\tilde{g}_{Z_i}}(Y)$ is the last vertex). Then $\tilde{g}_{Z_i}$ contributes a segment $\sigma_i^+ = \{(\tilde{g}_{Z_i}, z_i) \prec_p \cdots \prec_p (\tilde{g}_{Z_i}, \widetilde{\mathbf{T}}(\tilde{g}_{Z_i}))\}$. By the augmented version of [**20**, Corollary 4.11] (see Subsection 4.5), $\max \phi_{\tilde{g}_{Z_i}}(Y) = \max \phi_{\tilde{g}_{Z_i}}(X_{i-1})$, so $(\tilde{g}_{Z_{i-1}}, \widetilde{\mathbf{T}}(\tilde{g}_{Z_{i-1}})) \prec_p (\tilde{g}_{Z_i}, z_i)$. It follows that the union of the σ_i^+ are linearly ordered. Similarly, each σ_i^- has the form $\{(\tilde{g}_{X_i}, \widetilde{\mathbf{I}}(\tilde{g}_{X_i})) \prec_p \cdots \prec_p (\tilde{g}_{X_i}, x_i)\}$, where x_i is the last position before $\min \phi_{\tilde{g}_{X_i}}(Y)$. □

Let σ^+ be the concatenation of $\sigma_1^+ \cup \cdots \cup \sigma_n^+$ with the same linear order, and $\sigma^- = \sigma_m^- \cup \cdots \cup \sigma_1^-$. If both $\widetilde{\Sigma}^{\pm}(Y)$ are nonempty, then the $\tilde{g}_{X_0} = \tilde{g}_{Z_0}$ and $\phi_{\tilde{g}_{X_0}}(Y) = \emptyset$ by Theorem 4.6(2), so all its positions are in σ, and they follow and precede all pairs of σ_i^- and σ_i^+, respectively, for all $i > 0$. Denote the position on the top geodesic by σ^0.

The following lemma is the augmented analogue of [**20**, Lemma 6.1].

LEMMA 6.6 (Sigma projection). *There are constants M_1, M_2, depending only on S such that if $\widetilde{H}$ is any hierarchy and $Y \subset S$ is any subsurface, then*

$$\operatorname{diam}_Y(\pi_Y(\sigma^+(Y, \widetilde{H}))) \leqslant M_1 \quad \text{and} \quad \operatorname{diam}_Y(\pi_Y(\sigma^-(Y, \widetilde{H}))) \leqslant M_1.$$

Moreover, if Y is properly contained in the top domain of $\widetilde{\Sigma}(Y)$, then

$$\operatorname{diam}_Y(\pi_Y(\sigma(Y, \widetilde{H}))) \leqslant M_2.$$

Proof. The Bounded Geodesic Image Theorem 6.4 bounds $\operatorname{diam}_Y(\pi_Y(\sigma_i^{\pm}(Y)))$ and $\operatorname{diam}_Y(\pi_Y(\sigma^0))$ when Y is properly contained in the top domain. The transition from the last position of σ_i^+ to the first position of σ_{i+1}^+ involves adding disjoint curves, so it projects to a bounded step in $\mathcal{C}(Y)$ by Lemma 6.1; the same holds for other transitions in σ. Finally, the number of segments $\widetilde{\Sigma}^{\pm}(Y)$ contributes is bounded by $\xi(S) - \xi(Y)$. This completes the proof. □

6.3. *Large links*

In this subsection, we prove an augmented version of the Large Links Lemma 2.5. As with [**20**, Lemma 6.2], it follows almost immediately from Lemma 6.6.

LEMMA 6.7 (Large links for $\mathcal{AM}(S)$). *If $Y \subset S$ is any subsurface and $d_Y(\widetilde{\mathbf{I}}(\widetilde{H}), \widetilde{\mathbf{T}}(\widetilde{H})) > M_2$, then Y supports a geodesic $\tilde{g}_Y \in \widetilde{H}$. Conversely, if $\tilde{g}_Y \in \widetilde{H}$, then $||\tilde{g}_Y| - d_Y(\widetilde{\mathbf{I}}(\widetilde{H}), \widetilde{\mathbf{T}}(\widetilde{H}))| \leqslant 2M_1$.*

Proof. Let $\tilde{g}_{X_0} = \tilde{g}_{Z_0}$ be the top geodesic of $\widetilde{\Sigma}(Y)$. We have either $X_0 = Y$ or $Y \subsetneq X_0$. If the latter, then Y does not support a geodesic and Lemma 6.6 implies $d_Y(\widetilde{\mathbf{I}}(\widetilde{H}), \widetilde{\mathbf{T}}(\widetilde{H})) \leqslant \operatorname{diam}_Y(\pi_Y(\sigma)) \leqslant M_2$, proving the first statement.

For the second statement, if $Y = X_0$, then $\tilde{g}_Y = \tilde{g}_{X_0}$ by Theorem 4.6. Since σ^+ and σ^- contain both $\widetilde{\mathbf{T}}(\tilde{g}_Y), \widetilde{\mathbf{T}}(\widetilde{H})$ and $\widetilde{\mathbf{I}}(\tilde{g}_Y), \widetilde{\mathbf{I}}(\widetilde{H})$, respectively, Lemma 6.6 implies that

$$d_Y(\widetilde{\mathbf{I}}(\tilde{g}_Y), \widetilde{\mathbf{I}}(\widetilde{H})), d_Y(\widetilde{\mathbf{T}}(\tilde{g}_Y), \widetilde{\mathbf{T}}(\widetilde{H})) \leqslant M_1,$$

completing the proof. □

Let $M_5 = 2M_1 + 5$ and $M_6 = 4(M_1 + M_5 + 4)$, where M_1, M_2 are the constants from Lemmas 6.6 and 6.7, respectively. For any $\widetilde{\mu}, \widetilde{\eta} \in \mathcal{AM}(S)$ and augmented hierarchy $\widetilde{H}$ between them, set $\mathcal{G}_{M_6}(\widetilde{\mu}, \widetilde{\eta}) = \{\tilde{g}_Y \in \widetilde{H} \mid d_Y(\widetilde{\mu}, \widetilde{\eta}) > M_6\}$. Note that $|\mathcal{G}_{M_6}(\widetilde{\mu}, \widetilde{\eta})| = \sum_{\tilde{g}_Y \in \mathcal{G}_{M_6}(\widetilde{\mu}, \widetilde{\eta})} |\tilde{g}_Y|$ is independent of the choice of $\widetilde{H}$ up to coarse equality by Lemma 6.7.

Lemma 6.8. *There are constants $d_0, d_1 > 0$ dependent only on S such that $|\mathcal{G}_{M_6}(\widetilde{\mu}, \widetilde{\eta})| > d_0 \cdot |\widetilde{H}| - d_1$.*

Proof. As noted in the proof of [**20**, Theorem 6.10], the proof is an easy counting argument using the key fact that the number of component domains of any geodesic in $\widetilde{H}$ is a constant multiple of its length, where the constant only depends on S. □

6.4. *A distance formula for $\mathcal{AM}(S)$*

In this section, we derive a version of the Masur–Minsky distance formula for $\mathcal{AM}(S)$, which is related to Rafi's Theorem 2.10.

In [**20**], Masur–Minsky first related the size of a hierarchy to the sum of the size of its large links, and then used the $\mathcal{M}(S)$-analogue of Lemma 6.7 to obtain their distance formula. While this approach goes through to our setting, we first derive the distance formula and then relate it to augmented hierarchies via Lemma 6.7.

Theorem 6.9 (Distance formula for $\mathcal{AM}(S)$). *For each $K > M_6$, there are constants $C_1, C_2 > 0$, depending only on S and K such that, for any $\widetilde{\mu}, \widetilde{\eta} \in \mathcal{AM}(S)$, we have*

$$d_{\mathcal{AM}(S)}(\widetilde{\mu}, \widetilde{\eta}) \asymp_{(C_1, C_2)} \sum_{d_Y(\widetilde{\mu}, \widetilde{\eta}) > K} d_Y(\widetilde{\mu}, \widetilde{\eta}).$$

Proof. The second inequality follows from Proposition 5.7 and Lemma 6.8. For the first inequality, we adapt the hierarchy-free proof of the $\mathcal{M}(S)$-distance formula from the Aougab, Taylor and Webb [**28**].

Let $\widetilde{\mu}, \widetilde{\eta} \in \mathcal{AM}(S)$ and let $\widetilde{\mu} = \widetilde{\mu}_0, \ldots, \widetilde{\mu}_N = \widetilde{\eta}$ be any geodesic in $\mathcal{AM}(S)$ between them. Let $M = 10$ and $L = 4$ be the constants from Lemmata 6.3 and 6.1, respectively. Set $K = 5M + 3L$ and let $\mathcal{L}_K(\widetilde{\mu}, \widetilde{\eta}) = \{Y \mid d_Y(\widetilde{\mu}, \widetilde{\eta}) > K\}$ be the set of K large links for $\widetilde{\mu}$ and $\widetilde{\eta}$.

For each $Y \in \mathcal{L}_K(\widetilde{\mu}, \widetilde{\eta})$, let i_Y be the largest index k such that $d_Y(\widetilde{\mu}_0, \widetilde{\mu}_k) \leqslant 2M + L$ and t_Y be the smallest index j with $t_Y \geqslant i_Y$ such that $d_Y(\widetilde{\mu}_j, \widetilde{\mu}_N) \leqslant 2M + L$. Set $I_Y = [i_Y, t_Y] \subset \{0, 1, \ldots, N\}$ and observe that, since $d_Y(\widetilde{\mu}_i, \widetilde{\mu}_{i+1}) < L$ for each i by Lemma 6.2, the interval I_Y is well-defined. Since $d_Y(\widetilde{\mu}_0, \widetilde{\mu}_k), d_Y(\widetilde{\mu}_k, \widetilde{\mu}_N) \geqslant 2M + L$ for each $k \in I_Y$ by definition, it follows that $d_Y(\widetilde{\mu}_{i_Y}, \widetilde{\mu}_{t_Y}) \geqslant M + L$ by definition of K.

The following is essentially [**20**, Lemma 6.11], but the proof is from [**28**].

Lemma 6.10. *If $Y, Z \in \mathcal{L}_K(\widetilde{\mu}, \widetilde{\eta})$ and $Y \pitchfork Z$, then $I_Y \cap I_Z = \emptyset$.*

Proof of Lemma 6.10. The proof is an easy application of Lemma 6.3. Assume for a contradiction that there is a $k \in I_Y \cap I_Z$. Then Lemma 6.3 implies that either $d_Y(\partial Z, \widetilde{\mu}_0) \leqslant M$ or $d_Z(\partial Y, \widetilde{\mu}_0) \leqslant M$. Assume the former, since the proof in the latter case is the same.

Using the triangle inequality, we have

$$d_Y(\partial Z, \widetilde{\mu}_k) \geqslant d_Y(\widetilde{\mu}_0, \widetilde{\mu}_k) - d_Y(\widetilde{\mu}_0, \partial Z) \geqslant 2M + 1 - M \geqslant M + 1.$$

Thus Lemma 6.3 implies $d_Z(\partial Y, \widetilde{\mu}_k) \leqslant M$ so that

$$d_Z(\partial Y, \widetilde{\mu}_N) \geqslant d_Z(\widetilde{\mu}_k, \widetilde{\mu}_N) - d_Z(\widetilde{\mu}_k, \partial Y) \geqslant 2M + 1 - M \geqslant M + 1,$$

with Lemma 6.3 again implying that $d_Y(\partial Z, \widetilde{\mu}_N) \leqslant M$. Having assumed $d_Y(\partial Z, \widetilde{\mu}_0) \leqslant M$, we have

$$d_Y(\widetilde{\mu}_0, \widetilde{\mu}_N) \leqslant d_Y(\widetilde{\mu}_0, \partial Z) + d_Y(\partial Z, \widetilde{\mu}_N) \leqslant 2M < K$$

which contradicts the fact that $Y \in \mathcal{L}_K(\widetilde{\mu}, \widetilde{\eta})$, completing the proof of Lemma 6.10. □

Returning to the proof of Theorem 6.9, consider the collection $\{I_Y \mid Y \in \mathcal{L}_K(\widetilde{\mu}, \widetilde{\eta})\}$, which is a covering of $\{0, 1, \ldots, N\}$. Let $s = 2\xi(S) - 6$ be the number of pairwise nonoverlapping domains. By Lemma 6.10, each $k \in \{0, 1, \ldots, N\}$ is contained in at most s such I_Y. Thus

$$\sum_{Y \in \mathcal{L}_K(\widetilde{\mu}, \widetilde{\eta})} |I_Y| \leqslant s \cdot d_{\mathcal{AM}(S)}(\widetilde{\mu}, \widetilde{\eta}).$$

Applying Lemma 6.1, we have

$$d_Y(\widetilde{\mu}, \widetilde{\eta}) \leqslant d_Y(\widetilde{\mu}_{i_Y}, \widetilde{\mu}_{t_Y}) + 4M + 2L \leqslant L|I_Y| + 4M + 2L.$$

Since $d_Y(\widetilde{\mu}, \widetilde{\eta}) \geqslant 5M + 3L$ for each $Y \in \mathcal{L}_K(\widetilde{\mu}, \widetilde{\eta})$ by definition, it follows that $\frac{1}{5L} \cdot d_Y(\widetilde{\mu}, \widetilde{\eta}) \leqslant |I_Y|$. Combining all this, we get

$$\sum_{Y \in \mathcal{L}_K(\widetilde{\mu}, \widetilde{\eta})} d_Y(\widetilde{\mu}, \widetilde{\eta}) \leqslant 5\, sL \cdot d_{\mathcal{AM}(S)}(\widetilde{\mu}, \widetilde{\eta})$$

which completes the proof of the theorem. □

6.5. *Efficiency of augmented hierarchies*

The following is an immediate corollary of Theorem 6.9 and Lemmata 6.8 and 6.7.

THEOREM 6.11. *For each $K' > M_6$ there are constants $C_1', C_2' > 0$, depending only on S and K' such that, for any $\widetilde{\mu}, \widetilde{\eta} \in \mathcal{AM}(S)$ and augmented hierarchy $\widetilde{H}$ between them, we have*

$$\sum_{d_Y(\widetilde{\mu}, \widetilde{\eta}) > K'} d_Y(\widetilde{\mu}, \widetilde{\eta}) \asymp_{C_1', C_2'} |\widetilde{H}|.$$

Theorem 6.11 proves that augmented hierarchy paths are globally efficient. While their local efficiency can be proved using a subsurface projection argument well known to the experts, in Proposition A.4 of Appendix A, we prove that subpaths of augmented hierarchy paths are themselves augmented hierarchy paths in a natural way. Combining this with Theorem 6.11, we have the following corollary.

COROLLARY 6.12. *Augmented hierarchy paths are uniform quasigeodesics in $\mathcal{AM}(S)$.*

See Appendix A for more properties of hierarchy paths, augmented or otherwise.

7. $\mathcal{AM}(S)$ is quasiisometric to $\mathcal{T}(S)$

The goal of this section is the Main Theorem 7.13, which proves that $\mathcal{AM}(S)$ is quasiisometric to $\mathcal{T}(S)$ with the Teichmüller metric. We first make some estimates relating extremal length to curve graph distance, and then we define the maps between $\mathcal{AM}(S)$ and $\mathcal{T}(S)$. Finally, we prove that they are quasiisometries.

7.1. *Extremal length, intersection numbers, and curve complex distance*

In this subsection, we will show that two curves with bounded extremal length, with respect to one metric, have a bounded intersection number.

First, we will need the following useful result of Minsky.

Lemma 7.1 ([**21**, Lemma 5.1]). *For any $\sigma \in \mathcal{T}(S)$ and $\alpha, \beta \in \mathcal{C}(S)$, we have*

$$\mathrm{Ext}_\sigma(\alpha) \cdot \mathrm{Ext}_\sigma(\beta) \geqslant i_S(\alpha, \beta)^2.$$

Next, recall Kerckhoff's formula.

Theorem 7.2 ([**16**, Theorem 4]). *For any $\sigma_1, \sigma_2 \in \mathcal{T}(S)$,*

$$e^{2d_T(\sigma_1,\sigma_2)} = \sup_{\alpha \in \mathcal{C}(S)} \frac{\mathrm{Ext}_{\sigma_1}(\alpha)}{\mathrm{Ext}_{\sigma_2}(\alpha)}.$$

The following was observed by Rafi [**25**, Proposition 3.5].

Lemma 7.3. *For any $\sigma_1, \sigma_2 \in \mathcal{T}(S)$, if $\alpha, \beta \in \mathcal{C}(S)$ are such that $\mathrm{Ext}_{\sigma_1}(\alpha), \mathrm{Ext}_{\sigma_2}(\beta) \asymp 1$, then $\log i_S(\alpha, \beta) \prec d_T(\sigma_1, \sigma_2)$. In particular, if $d_T(\sigma_1, \sigma_2) \asymp 1$, then $i_S(\alpha, \beta) \asymp 1$.*

Proof. The proof is an easy application of Lemma 7.1 and Theorem 7.2:

$$i_S(\alpha, \beta) \leqslant \mathrm{Ext}_{\sigma_1}(\alpha) \cdot \mathrm{Ext}_{\sigma_1}(\beta) \leqslant \mathrm{Ext}_{\sigma_1}(\alpha) \cdot \mathrm{Ext}_{\sigma_2}(\beta) e^{2d_T(\sigma_1,\sigma_2)}.$$

Since $\mathrm{Ext}_{\sigma_1}(\alpha), \mathrm{Ext}_{\sigma_2}(\beta) \asymp 1$, applying log to both sides gives the first conclusion, which is easily seen to apply the second conclusion. We note that the bounds on extremal length determine the bounds on intersection number. □

We will also use the following well-known estimate relating curve complex distance to intersection number.

Lemma 7.4. *For any $\alpha, \beta \in \mathcal{C}(S)$, we have $d_{\mathcal{C}(S)}(\alpha, \beta) \prec i_S(\alpha, \beta)$.*

Proof. When $\xi(S) > 4$, this is [**19**, Lemma 2.1]. When $\xi(S) = 4$, then this is an easy argument in the Farey graph. When S is an annulus or horoball, this follows from arguments in [**20**, Subsection 2.4]. □

Combining these ideas, we have the following proposition.

Proposition 7.5. *Let $\sigma_1, \sigma_2 \in \mathcal{T}(S)$ be such that $d_T(\sigma_1, \sigma_2) \asymp 1$. For any $\alpha, \beta \in \mathcal{C}(S)$ with $\mathrm{Ext}_{\sigma_1}(\alpha), \mathrm{Ext}_{\sigma_2}(\beta) \asymp 1$ and $Y \subset S$ such that $\pi_Y(\alpha), \pi_Y(\beta) \neq \emptyset$, we have $d_Y(\alpha, \beta) \asymp 1$.*

Proof. Since $\pi_Y(\alpha), \pi_Y(\beta) \neq \emptyset$, $d_Y(\alpha,\beta)$ is defined, and Lemmata 7.3 and 7.4 imply that

$$d_Y(\alpha,\beta) \prec i_Y(\alpha,\beta) \prec i_S(\alpha,\beta) \asymp 1,$$

completing the proof. □

7.2. *From $\mathcal{T}(S)$ to $\mathcal{AM}(S)$*

We are now ready to define maps between $\mathcal{AM}(S)$ and $\mathcal{T}(S)$, which we later prove, are quasiisometries in Theorem 7.13.

Let $\alpha \in \mathcal{C}(S)$ and $\sigma \in \mathcal{T}(S)$. Define a map $d_\alpha : \mathcal{T}(S) \to \mathbb{Z}_{\geqslant 0}$ by

$$d_\alpha(\sigma) = \begin{cases} \max\left\{k \,\middle|\, \frac{\epsilon_0}{2^{k+1}} < \mathrm{Ext}_\sigma(\alpha) < \frac{\epsilon_0}{2^k}\right\} & \text{if } \mathrm{Ext}_\sigma(\alpha) < \epsilon_0, \\ 0 & \text{if } \mathrm{Ext}_\sigma(\alpha) \geqslant \epsilon_0. \end{cases}$$

For each $\sigma \in \mathcal{T}(S)$, let μ_σ be any marking such that base(μ_σ) is a Bers pants decomposition for σ, as in Theorem 2.11, and so that we have chosen traversals to base(μ_σ) to minimize l_σ. Note that there may be finitely many choices of transversals for each base curve, and thus finitely many such markings μ_σ.

Define $F : \mathcal{T}(S) \to \mathcal{AM}(S)$ by $F(\sigma) = (\mu_\sigma, d_{\alpha_1}(\sigma), \ldots, d_{\alpha_n}(\sigma))$ where base$(\mu_\sigma) = \{\alpha_1, \ldots, \alpha_n\}$. We think of F as choosing a *shortest augmented marking* for each $\sigma \in \mathcal{T}(S)$, and outside the context of the map F, we may write $\widetilde{\mu}_\sigma$ for a shortest augmented marking for a point $\sigma \in \mathcal{T}(S)$. The following lemma proves that F is coarsely well-defined.

Lemma 7.6. *For any $\sigma \in \mathcal{T}(S)$, we have* $\mathrm{diam}_{\mathcal{AM}(S)}(F(\sigma)) \asymp 1$.

Proof. Let $\sigma \in \mathcal{T}(S)$ and let $\widetilde{\mu}_\sigma, \widetilde{\mu}'_\sigma \in F(\sigma) \subset \mathcal{AM}(S)$. Recall from Lemma 2.13 that $\mathrm{Ext}_\sigma(\alpha) < L_0$ for each $\alpha \in \mathrm{base}(\widetilde{\mu}_\sigma) \cup \mathrm{base}(\widetilde{\mu}'_\sigma)$, where L_0 depends only on S. The goal is to bound all subsurface projections between $\widetilde{\mu}_\sigma$ and $\widetilde{\mu}'_\sigma$, allowing us to invoke the distance formula, Theorem 6.9.

Let $Y \subset S$ be nonannular. If there are not $\alpha \in \mathrm{base}(\widetilde{\mu}_\sigma)$ and $\beta \in \mathrm{base}(\widetilde{\mu}'_\sigma)$ with $i_S(\alpha,\beta) > 0$ and $\pi_Y(\alpha), \pi_Y(\beta) \neq \emptyset$, then clearly $d_Y(\widetilde{\mu}_\sigma, \widetilde{\mu}'_\sigma) < 4$ by Lemma 6.1. If there are such and, then since $\mathrm{Ext}_\sigma(\alpha), \mathrm{Ext}_\sigma(\beta) < L_0$, it follows from Proposition 7.5 that $d_Y(\alpha,\beta) \asymp 1$, with Lemma 6.1 implying $d_Y(\widetilde{\mu}_\sigma, \widetilde{\mu}'_\sigma) \asymp 1$.

Now let $\gamma \in \mathcal{C}(S)$ be any curve. If $\gamma \notin \mathrm{base}(\widetilde{\mu}_\sigma) \cup \mathrm{base}(\widetilde{\mu}'_\sigma)$, then Proposition 7.5 implies that $d_\gamma(\widetilde{\mu}_\sigma, \widetilde{\mu}'_\sigma)$ is uniformly bounded. Since $D_\gamma(\widetilde{\mu}_\sigma) = D_\gamma(\widetilde{\mu}'_\sigma) = 0$, we can conclude that $d_{\mathcal{H}(\gamma)}(\widetilde{\mu}_\sigma, \widetilde{\mu}'_\sigma) \asymp 1$. If $\gamma \in \widetilde{\mu}_\sigma \cap \widetilde{\mu}'_\sigma$, then $d_{\mathcal{H}(\gamma)}(\widetilde{\mu}_\sigma, \widetilde{\mu}'_\sigma) \asymp 1$ by definition.

Finally, if $\gamma \in \widetilde{\mu}_\sigma$ but $\gamma \notin \widetilde{\mu}_\sigma$, then $l_\sigma(\gamma) > \epsilon_0$ by Lemma 2.13. It follows then the length of the shortest transverse curve to γ, t_γ, has $l_\sigma(t_\gamma)$ uniformly bounded, with the Collar Lemma implying that $\mathrm{Ext}_\sigma(t_\gamma)$ is uniformly bounded. Since $\gamma \notin \widetilde{\mu}'_\sigma$, there is a $\gamma' \in \mathrm{base}(\widetilde{\mu}'_\sigma)$ with $i_S(\gamma,\gamma') > 0$. Since $\mathrm{Ext}_\sigma(\gamma') < L_0$, we can then apply the above intersection number argument to derive that $d_{\mathcal{H}(\gamma)}(\widetilde{\mu}_\sigma, \widetilde{\mu}'_\sigma) \asymp 1$. □

7.3. *From $\mathcal{AM}(S)$ to $\mathcal{T}(S)$*

We now construct an embedding $G : \mathcal{AM}(S) \to \mathcal{T}(S)$ in terms of Fenchel–Nielsen coordinates. Consider an augmented marking $\widetilde{\mu} \in \mathcal{AM}(S)$ with $\widetilde{\mu} = (\mu, D_{\alpha_1}, \ldots, D_{\alpha_n})$. In building coordinates for $G(\widetilde{\mu})$, we are given a clear choice of a pants decomposition, base(μ), and bounds for the length coordinates, $\epsilon_0/2^{D_{\alpha_i}+2} < l_{\alpha_i} < \epsilon_0/2^{D_{\alpha_i}+1}$. Given a choice of length coordinates, say $l_{\alpha_i} = \epsilon_0/2^{D_{\alpha_i}+3/2}$, we can use the transverse curve data (α_i, t_i) to pick out a unique twisting number, $\tau_{\alpha_i}(t_i)$, and thus a unique metric on S, as follows.

For each i, α_i either bounds one or two pairs of pants, depending on whether α_i lives in a four-holed sphere or a one-holed torus. As we have chosen lengths for all the curves in the pants decomposition, the metrics on the pairs of pants are uniquely determined.

In the case of the four-holed sphere, consider the two unique essential geodesic arcs, β_1, β_2 in the pairs of pants connecting α_i to itself. Let $\tau_{\alpha_i}(t_i)$ be the unique twisting number associated to the gluing of the pairs of pants at α_i which connects β_1 to β_2 to realize t_i.

Similarly, for the case when α_i bounds two cuffs on one pair of pants which glue into a one-holed torus, there is a unique geodesic arc, β, connecting the two copies of α_i. Let $\tau_{\alpha_i}(t_i)$ be the unique twisting number associated to the gluing of the copies of α_i which connected the two ends of β to realize t_i.

We can now define $G : \mathcal{AM}(S) \to \mathcal{T}(S)$ by $G(\widetilde{\mu}) = (l_{\alpha_i}, \tau_{\alpha_i}(t_i))_i$. Since G sends each augmented marking to a unique point for which each curve in the base of that marking is short, the shortest augmented marking for any point in the image of G is unambiguous by Lemma 2.13; that is, $F \circ G(\widetilde{\mu}) = \widetilde{\mu}$. Thus we obtain the following lemma.

LEMMA 7.7. *$F \circ G = id_{\mathcal{AM}(S)}$; in particular, G is an embedding and F is a surjection.*

7.4. *The quasiisometry*

We prove, in a series of lemmata, that G is a quasiisometry by showing that F and G satisfy the conditions of the following elementary lemma.

LEMMA 7.8. *Let X and Y be metric spaces. If $g : X \to Y$ and $f : Y \to X$ are both L-lipschitz, and there exists a $K > 0$ such that $d_X(f(g(x)), x) < K$ for each $x \in X$, then g is a $(L, 2LK)$-quasiisometric embedding. If $g(X) \subset Y$ is also quasidense, then g is a quasiisometry.*

Proof. Let $x_1, x_2 \in X$. Then

$$d_X(x_1, x_2) < L \cdot d_Y(g(x_1), g(x_2)) < L^2 \cdot d_X(f(g(x_1)), f(g(x_2))) \leqslant L^2 d_X(x_1, x_2) + 2L^2 K$$

with the triangle inequality implying the last inequality. Dividing everything by L completes the proof. □

We begin by proving that F is lipschitz, the proof of which proceeds similarly to Lemma 7.6.

LEMMA 7.9. *There is a constant $L_2 = L_2(S) > 0$ such that, for any $\sigma_1, \sigma_2 \in \mathcal{T}(S)$ with $d_T(\sigma_1, \sigma_2) = 1$, $d_{\mathcal{AM}(S)}(\widetilde{\mu}_{\sigma_1}, \widetilde{\mu}_{\sigma_2}) < L_2$.*

Proof. Suppose that $\sigma_1, \sigma_2 \in \mathcal{T}(S)$ with $d_T(\sigma_1, \sigma_2) = 1$. We will uniformly bound all subsurface projections between $\widetilde{\mu}_{\sigma_1}$ and $\widetilde{\mu}_{\sigma_2}$. The result will then follow from the distance formula, Theorem 6.9.

Let $Y \subset S$ be any nonannular subsurface and let $\alpha \in \mathrm{base}(\widetilde{\mu}_{\sigma_1}), \beta \in \mathrm{base}(\widetilde{\mu}_{\sigma_2})$ with $\pi_Y(\alpha), \pi_Y(\beta) \neq \emptyset$. By definition of F and Lemma 2.13, we have $\mathrm{Ext}_{\sigma_1}(\alpha), \mathrm{Ext}_{\sigma_2}(\beta) \asymp 1$ for any $\alpha \in \mathrm{base}(\widetilde{\mu}_{\sigma_1}), \beta \in \mathrm{base}(\widetilde{\mu}_{\sigma_2})$. It then follows from Proposition 7.5 and Lemma 6.1 that $d_Y(\widetilde{\mu}_{\sigma_1}, \widetilde{\mu}_{\sigma_2}) \asymp 1$.

It remains to bound projections in horoballs. Let $\alpha \in \mathcal{C}(S)$ and note that $D_\alpha(\widetilde{\mu}_{\sigma_1}) \asymp D_\alpha(\widetilde{\mu}_{\sigma_2})$ by definition and Theorem 2.8, because $d_T(\sigma_1, \sigma_2) = 1$. It will thus suffice to bound projections to annular complexes. There are four cases, depending on whether $\alpha \in \mathrm{base}(\widetilde{\mu}_{\sigma_i})$ for each i.

If $\alpha \notin \mathrm{base}(\widetilde{\mu}_{\sigma_1}) \cup \mathrm{base}(\widetilde{\mu}_{\sigma_2})$, then there are curves $\beta \in \mathrm{base}(\widetilde{\mu}_{\sigma_1})$ and $\gamma \in \mathrm{base}(\widetilde{\mu}_{\sigma_2})$ with $i_S(\alpha, \beta), i_S(\alpha, \gamma) > 0$. Thus Proposition 7.5 and Lemma 6.1 imply that $d_\alpha(\widetilde{\mu}_{\sigma_1}, \widetilde{\mu}_{\sigma_2}) \asymp 1$, as required.

Now suppose that $\alpha \in \mathrm{base}(\widetilde{\mu}_{\sigma_1}) \cup \mathrm{base}(\widetilde{\mu}_{\sigma_2})$. Since $d_T(\sigma_1, \sigma_2) = 1$, Theorem 2.8 implies there exist constants $C', D' > 0$, depending only on S such that if $\min\{D_\alpha(\widetilde{\mu}_x) D_\alpha(\widetilde{\mu}_y)\} > C'$, then $d_{\mathcal{H}(\alpha)}(\widetilde{\mu}_x, \widetilde{\mu}_y) < D'$.

If not, then $\mathrm{Ext}_{\sigma_1}(\alpha)$ and $\mathrm{Ext}_{\sigma_2}(\alpha)$ are uniformly bounded above and below. Thus there exist curves $\beta_1, \beta_2 \in \mathcal{C}(S)$ with $i_S(\beta_i, \alpha) > 0$ and $\mathrm{Ext}_{\sigma_i}(\beta_i) \asymp 1$ for $i = 1, 2$. Since the length of α is uniformly bounded below in both σ_1 and σ_2, it follows that the shortest transverse curves to α in σ_1, σ_2 must have uniformly bounded twisting around α relative to β_1, β_2 in σ_1, σ_2, respectively.

For $i = 1, 2$, if $\alpha \in \mathrm{base}(\widetilde{\mu}_{\sigma_i})$, then let $t_{\alpha,i}$ be its transversal. The above argument then implies that $d_\alpha(\beta_i, t_{\alpha,i}) \asymp 1$. If $\alpha \in \mathrm{base}(\widetilde{\mu}_{\sigma_1}) \cap \mathrm{base}(\widetilde{\mu}_{\sigma_2})$, then the triangle inequality implies that $d_\alpha(\widetilde{\mu}_{\sigma_1}, \widetilde{\mu}_{\sigma_2}) \asymp 1$.

If $\alpha \in \mathrm{base}(\widetilde{\mu}_{\sigma_1})$ but $\alpha \notin \mathrm{base}(\widetilde{\mu}_{\sigma_2})$, then there is some curve $\gamma \in \mathrm{base}(\widetilde{\mu}_{\sigma_2})$ with $i_S(\gamma, \alpha) > 0$, and since $\mathrm{Ext}_{\sigma_2}(\gamma) \asymp 1$, Proposition 7.5 applied to γ and β_1 implies that $d_\alpha(\widetilde{\mu}_{\sigma_1}, \widetilde{\mu}_{\sigma_2}) \asymp 1$. This completes the proof. □

Next, we prove that G is lipschitz.

Lemma 7.10. *There is a constant $L_1 = L_1(S) > 0$ such that, for any $\widetilde{\mu}_1, \widetilde{\mu}_2 \in \mathcal{AM}(S)$ adjacent vertices in $\mathcal{AM}(S)$, $d_{\mathcal{T}(S)}(G(\widetilde{\mu}_1), G(\widetilde{\mu}_2)) < L_1$.*

Proof. Let $\epsilon > 0$ be as in Theorem 2.8. First, suppose that $\widetilde{\mu}_1$ and $\widetilde{\mu}_2$ differ by a vertical edge or horizontal edge in a horoball, $\mathcal{H}_\alpha$, where $\alpha \in \mathrm{base}(\widetilde{\mu}_1) \cap \mathrm{base}(\widetilde{\mu}_2)$. Recall that the length of α in both $G(\widetilde{\mu}_1)$ and $G(\widetilde{\mu}_2)$ is less than ϵ by the definition of G. By Minsky's Theorem 2.8, $G(\widetilde{\mu}_1)$ and $G(\widetilde{\mu}_2)$ coarsely live in the product $\mathbb{H}_\alpha \times \mathcal{T}(S \setminus \alpha)$. The projections of $G(\widetilde{\mu}_1)$ and $G(\widetilde{\mu}_2)$ to $\mathcal{T}(S \setminus \alpha)$ are identical, so $d_T(G(\widetilde{\mu}_1), G(\widetilde{\mu}_2))$ is (up to an additive constant) equal to the distance in $\mathbb{H}_\alpha$ of the projections of $G(\widetilde{\mu}_1)$ and $G(\widetilde{\mu}_2)$ to $\mathbb{H}_\alpha$, again by Minsky's Theorem 2.8. This distance is coarsely the corresponding distance in a horodisk, via Proposition 3.2, which is precisely 1 by Lemma 7.7. Thus there is a uniform bound on $d_T(G(\widetilde{\mu}_1), G(\widetilde{\mu}_2))$.

Now suppose that $\widetilde{\mu}_1$ and $\widetilde{\mu}_2$ differ by a flip move, so that they only differ in their underlying marking. Then, as argued in [**25**, Lemma 5.6], there are only finitely many pairs of such markings up to homeomorphism, and the result follows from the local finiteness of $\mathcal{AM}(S)$, Lemma 3.5. □

Finally, we prove that $G(\mathcal{AM}(S)) \subset \mathcal{T}(S)$ is quasidense, but before we do so, we need the following lemma.

Lemma 7.11. *Every point in the ϵ_0-thick part of $\mathcal{T}(S)$ is a uniformly bounded distance away from the ϵ-thin parts of $\mathcal{T}(S)$. This bound depends only on the topology of S.*

Proof. If $\sigma \in \mathcal{T}(S)$ is in the ϵ_0-thick part of $\mathcal{T}(S)$ and $\mu_\sigma \in \mathcal{M}(S)$ is the shortest marking for σ with $\mathrm{base}(\mu_\sigma) = \{\gamma_1, \dots, \gamma_n\} = \gamma \in \mathcal{C}(S)$, then there is a uniform upper bound on the length of the γ_i, which depends only on the topology of S. Thus there is a uniform bound on the distance between σ and some point $\sigma_{thin} \in Thin_\gamma$, which is obtained by scaling the lengths of the curves in γ in σ to be less than ϵ_0. In fact, this holds for points in the ϵ_0-thick part of $\mathcal{T}(Y)$ for every subsurface $Y \subset S$, with the same constant bounding the distance to a uniformly thin part. □

Lemma 7.12. *$G(\mathcal{AM}(S))$ is quasidense in $\mathcal{T}(S)$.*

Proof. We show by induction that $G(\mathcal{AM}(S))$ is quasidense in the ϵ_0-thin parts of $\mathcal{T}(S)$. Let $\sigma \in \mathcal{T}(S)$ and let $F(\sigma) = \widetilde{\mu}_\sigma \in \mathcal{AM}(S)$ a shortest augmented marking for σ. It suffices to show that there is a uniform bound on the distance between σ and $G(\widetilde{\mu}_\sigma)$. Suppose first that $\sigma \in Thin_\gamma$ where $\gamma = \{\gamma_1, \ldots, \gamma_n\} \subset \mathcal{C}(S)$ is a maximal simplex, that is, pants decomposition, of S. Then by Theorem 2.8, σ and $G(\widetilde{\mu}_\sigma)$ coarsely live in $\prod_i \mathbb{H}_{\gamma_i}$ and have length coordinates which differ at most by $\frac{\epsilon_0}{2}$. As there is a uniform bound on the distance in each $\mathbb{H}_{\gamma_i}$ and on the dimension of the simplex γ, it follows that σ and $G(\widetilde{\mu}_\sigma)$ are uniformly close.

Now suppose that $\sigma \in Thin_\gamma$, where $\gamma = \{\gamma_1, \ldots, \gamma_{n-1}\} \subset \mathcal{C}(S)$ is a simplex of dimension one less than maximal. Then σ and $G(\widetilde{\mu}_\sigma)$ coarsely live in $\prod_i \mathbb{H}_{\gamma_i} \times \mathcal{T}(S \setminus \gamma)$. If μ_σ is the shortest marking for σ, with $\mathrm{base}(\mu_\sigma) = \{\gamma_1, \ldots, \gamma_{n-1}, \alpha\}$, then α was the shortest curve in σ in $\mathcal{C}(S \setminus \gamma)$ and $G(\widetilde{\mu}_\sigma)$ lives in $\prod_i \mathbb{H}_{\gamma_i} \times \mathbb{H}_\alpha$. By Lemma 7.11, there is a uniform bound on the distance between $\pi_{\mathcal{T}(S\setminus\gamma)}(\sigma)$ and $Thin_\alpha \subset \mathcal{T}(S \setminus \gamma)$. Thus there is a uniform bound on the distance between σ and $Thin_{\gamma\cup\{\alpha\}} \subset \mathcal{T}(S)$ by Theorem 2.8. Since $G(\mathcal{AM}(S))$ is quasidense in $Thin_{\gamma\cup\{\alpha\}}$, it follows by induction that $G(\mathcal{AM}(S))$ is quasidense in $\mathcal{T}(S)$, completing the proof. □

Combining Lemmata 7.10, 7.9, and 7.7 with Lemma 7.8, we have the following theorem.

THEOREM 7.13. *$\mathcal{AM}(S)$ with the path metric is quasi-isometric to $\mathcal{T}(S)$ with the Teichmüller metric.*

As an application of Theorems 7.13 and 6.9, we have a new proof of Rafi's distance formula for $\mathcal{T}(S)$.

THEOREM 7.14 (A distance formula for $\mathcal{T}(S)$). *There exists a $K' = K'(S) > 0$ such that, for any $\sigma_1, \sigma_2 \in \mathcal{T}(S)$ with shortest augmented markings $\widetilde{\mu}_{\sigma_1}, \widetilde{\mu}_{\sigma_2} \in \mathcal{AM}(S)$, we have*

$$d_{\mathcal{T}(S)}(\sigma_1, \sigma_2) \asymp \sum_{d_Y(\widetilde{\mu}_{\sigma_1}, \widetilde{\mu}_{\sigma_2}) > K} d_Y(\widetilde{\mu}_{\sigma_1}, \widetilde{\mu}_{\sigma_2}) + \sum_{d_{\mathcal{H}(\alpha)}(\widetilde{\mu}_{\sigma_1}, \widetilde{\mu}_{\sigma_2}) > K} d_{\mathcal{H}(\alpha)}(\widetilde{\mu}_{\sigma_1}, \widetilde{\mu}_{\sigma_2})$$

where the $Y \subset S$ are nonannular.

Appendix. *Hierarchical technicalities*

In this appendix, we prove a number of technical results about hierarchies. Perhaps the main goal is to prove that subpaths of hierarchy paths are hierarchy paths in a natural way. We also analyze special subsegments of hierarchy paths during which progress through a subsurface is made. We have sequestered this section from the rest of the paper to enhance the coherence of the main exposition. We hope that some of these results will be of independent interest.

In order to minimize notational clutter, we will work with standard hierarchies, but everything holds *mutatis mutandis* for augmented hierarchies and hierarchies without annuli.

A.1. *Active segments*

In this subsection, we introduce the notion of an active segment of a subsurface along a hierarchy path.

For any geodesic $g_Y \in H$, let $v_{Y,\mathrm{int}}$ and $v_{Y,\mathrm{ter}}$ be the initial and terminal vertices of g_Y, respectively. By [**23**, Lemma 5.8], any resolution of slices of H contains slices with $(g_Y, v_{Y,\mathrm{int}})$ and $(g_Y, v_{Y,\mathrm{ter}})$. Fixing a hierarchy path Γ, let $\tau_{g_Y,\mathrm{int}}$ and $\tau_{g_Y,\mathrm{ter}}$ be the first and last such slices, respectively, along the resolution of slices which gives Γ, with $\mu_{Y,\mathrm{int}}$ and $\mu_{Y,\mathrm{ter}}$ their respective augmented markings.

We call $\Gamma_Y = [\mu_{Y,\mathrm{int}}, \mu_{Y,\mathrm{ter}}] \subset \Gamma$ the *active segment* of Y along Γ. It is clear from the definition of an elementary move of augmented slices that Γ_Y is contiguous. We remark that our notion of active segment is similar to Brock–Canary–Minsky's intervals from [8, p. 64] and Rafi's notion of active interval of a Teichmüller geodesic from [26].

See Subsection A.4 for a structural result about active segments.

A.2. *Subordinancy and slices*

Let H be any hierarchy between $\mu, \eta \in \mathcal{M}(S)$ and let Γ be any hierarchy path based on H. Let $g_Y \in H$ and recall from Subsection A.1 the definition of an active segment of Y along Γ, namely Γ_Y. The following lemma connects direct subordinancy for $g_Y \in H$ to the initial and terminal slices of Γ_Y, $\tau_{Y,\mathrm{int}}$, and $\tau_{Y,\mathrm{ter}}$, respectively.

LEMMA A.1 (Subordinancy and slices). *Let H and Γ be as above. Let $g_X, g_Y \in H$ with $D(g_X) = X$, and $D(g_Y) = Y$ with Y a component domain of (g_X, x). Then $(g_X, x) \in \tau_{Y,\mathrm{int}}$ if and only if $g_X \swarrow g_Y$. Similarly, $(g_X, x) \in \tau_{Y,\mathrm{ter}}$ if and only if $g_Y \searrow g_X$.*

Proof. We prove the direct backward subordinate case, as the direct forward subordinate case is similar. We first prove the forward implication.

The proof involves understanding what happens in the transition into the initial slice of Γ_Y. Let $\mu \in \Gamma$ be the marking preceding $\mu_{Y,\mathrm{int}}$ along Γ, and let $\tau_\mu \to \tau_{Y,\mathrm{int}}$ be the slices in the resolution of H which gives Γ. Since $\tau_\mu \to \tau_{Y,\mathrm{int}}$ is an elementary move of slices, there are by definition some geodesic $g_W \in H$ and vertices $w, w' \in g_W$ so that $\tau_\mu \to \tau_{Y,\mathrm{int}}$ is essentially realizing the transition from w to w' along g_W. The reorganization of the hierarchical data is contained in the transition slices $\sigma \subset \tau_\mu$ and $\sigma' \subset \tau_{Y,\mathrm{int}}$ with $\tau_\mu \setminus \sigma = \tau_{Y,\mathrm{int}} \setminus \sigma'$. We shall find g_Y and g_X in these transition slices.

Let $y_{\mathrm{int}} \in g_Y$ be the initial vertex of g_Y. By assumption, and the fact that $\tau_\mu \setminus \sigma = \tau_{Y,\mathrm{int}} \setminus \sigma'$, we must have that $(g_Y, y_{\mathrm{int}}) \in \sigma'$, as $\tau_{Y,\mathrm{int}}$ is the first slice involving g_Y. This implies by definition of σ' that $w|_Y \neq \emptyset$. Property (S3) of slices implies there is a pair $(g_X, x) \in \sigma'$, where $g_X \in H$ with $D(g_X) = X$ and Y a component domain of (X, x). Consider the simple case where $g_X \swarrow Y$; in order to conclude that $g_X \swarrow g_Y$, we need to prove $\mathbf{I}(g_Y) = \mathbf{I}(Y, g_X)$. Applying [20, Theorem 4.7(1)], there exists $g_Z \in H$ with $g_Y \searrow g_Z$, which implies that $Y \searrow g_Z$. Part (H2) of the definition of a hierarchy implies there is $g'_Y \in H$ with $g_X \swarrow g'_Y \searrow g_Z$, but [20, Theorem 4.7(4)] states that geodesics in H are uniquely determined by their domains, so $g'_Y = g_Y$ and $g_X \swarrow g_Y$.

In the general case, $\mathbf{I}(Y, g_X) = \mathbf{I}(g_X)|_Y$, and we do not know what the latter marking is. We will in fact show that $\mathbf{I}(g_X) = w'|_X$, but this requires an inductive application of the above argument. To begin, property (S3) of slices implies there is a sequence of pairs $\{(g_{X_i}, x_i)\}_{i=1}^n$ in σ' with $(g_{X_1}, x_1) = (g_Y, Y)$, $(g_{X_2}, x_2) = (g_X, x)$, and $(g_{X_n}, x_n) = (g_W, w')$ such that, for each $1 \leqslant i < n$, X_i is a component domain of (X_{i+1}, x_{i+1}); moreover, the definition of σ' implies that x_i is the initial vertex of g_{X_i} when $1 \leqslant i < n$. Since w' is not the initial vertex of g_W, we have $\mathbf{I}(X_{n-1}, g_{X_n}) = w|_{X_{n-1}}$, which is nonempty by definition of σ'. Moreover, since H is a hierarchy, [20, Theorem 4.7(1)] implies that there is some $g_{Z_n} \in H$ with $g_{X_{n-1}} \searrow g_{Z_n}$, implying that $X_{n-1} \searrow g_{Z_n}$. The definition of a hierarchy then implies that there is $g'_{X_{n-1}} \in H$ with $D(g'_{X_{n-1}}) = X_{n-1}$ and $g_W \swarrow g'_{X_{n-1}} \searrow g_{Z_n}$. But [20, Theorem 4.7(4)] implies that geodesics in H are uniquely determined by their domains, so $g'_{X_{n-1}} = g_{X_{n-1}}$ and $g_W \swarrow g_{X_{n-1}}$, implying $\mathbf{I}(g_{X_{n-1}}) = \mathbf{I}(X_{n-1}, g_W) = w'|_{X_n}$.

Now considering $g_{X_{n-1}}$, x_{n-1} is its initial vertex, so $\mathbf{I}(X_{n-2}, g_{X_{n-1}}) = \mathbf{I}(g_{X_{n-1}})|_{X_{n-2}} = (w'|_{X_{n-1}})|_{X_{n-2}} = w'|_{X_{n-2}}$, which is nonempty by definition of σ'. This implies that $g_{X_{n-1}} \swarrow X_{n-2}$. Proceeding as above, we find a $g_{Z_{n-1}} \in H$ with $g_{X_{n-2}} \searrow g_{Z_{n-1}}$ and, as before, we can conclude that $g_{X_{n-1}} \swarrow g_{X_{n-2}}$, implying that $\mathbf{I}(g_{X_{n-2}}) = \mathbf{I}(X_{n-2}, g_{X_{n-1}}) = w'|_{X_{n-2}}$. Proceeding by induction, we see, for $1 \leqslant i < n$, that $g_{X_{i+1}} \swarrow g_{X_i}$ and $\mathbf{I}(g_{X_i}) = w'|_{X_i}$. In particular, $g_X \swarrow g_Y$ and $\mathbf{I}(g_Y) = w'|_Y$, which completes the proof of the forward implication.

For the reverse implication, suppose that $g_X \swarrow g_Y$. Since Y has a nonempty active segment, there exist a domain $X' \subset S$, a geodesic $g_{X'} \in H$ with $D(g_{X'}) = X'$, and a vertex $x' \in g_{X'}$ with Y a component domain of $(g_{X'}, x')$ such that $(g_{X'}, x') \in \tau_{Y,\mathrm{int}}$. The first part of this lemma implies then that $g_{X'} \swarrow g_Y$, while our assumption is that $g_X \swarrow g_Y$; [**20**, Theorem 4.7] implies that $g_{X'} = g_X$, and thus $x' = x$, completing the proof. □

A.3. *Subpaths of hierarchy paths*

In this subsection, we prove that subpaths of hierarchy paths are themselves hierarchy paths in a natural way.

Truncating hierarchies. Let H be any hierarchy between $\mu, \eta \in \mathcal{M}(S)$, Γ be a hierarchy path based on H, and $[\mu_0, \eta_0] \subset \Gamma$ be any subpath. We will define a way to truncate the geodesics in H to their relevant contributions to $[\mu_0, \eta_0]$. Initial and terminal marking data are then inductively added to the truncated geodesics. In Lemma A.2, we prove that the resulting collection, H_0, is a hierarchy. We then show in Lemma A.3 that the original slice resolution of H from which Γ was obtained is a slice resolution for H_0. We immediately obtain that $[\mu_0, \eta_0]$ is a hierarchy path based on H_0 in Proposition A.4.

Let $g_Y \in H$ with $D(g_Y) = Y$. Suppose $g_Y \in H$ is such that $\Gamma_Y \cap [\mu_0, \eta_0] \neq \emptyset$. We can form a new geodesic $g'_Y \subset \mathcal{C}(Y)$ as follows: If $\mu_0 \in \Gamma_Y$ with τ_{μ_0} the corresponding slice, then there exists a pair $(g_Y, v_{Y,\mu_0}) \in \tau_{\mu_0}$, and we can remove the (possibly empty) initial segment of g_Y to obtain a geodesic g'_Y with initial vertex v_{Y,μ_0}; we similarly truncate the end segment of g_Y if it contributes to a pair in τ_{η_0}. If $\mu_0 \in \Gamma_Y$, then we say Γ_Y is *initially truncated* by $[\mu_0, \eta_0]$; similarly, if $\eta_0 \in \Gamma_Y$, then we say that Γ_Y is *terminally truncated* by $[\mu_0, \eta_0]$. We note that v_{Y,μ_0} and v_{Y,η_0} can be the initial and terminal vertices of g_Y, respectively. If $\Gamma_Y \subset [\mu_0, \eta_0]$, set $g'_Y = g_Y$.

Building the initial and terminal markings. Let $H' = \{g'_Y \mid \Gamma_Y \cap [\mu_0, \eta_0] \neq \emptyset\}$. In order to complete H' into a collection of tight geodesics, we need to attach initial and terminal marking data to the g'_Y. We only describe how to build initial marking data, as terminal marking data are built similarly. For the initial marking data, the key is determining which geodesic in H_0 each g'_Y should be directly backward subordinate, and there are two cases. First, suppose that Γ_Y is initially truncated. We can build $\mathbf{I}(g'_Y)$ inductively from μ_0 as follows.

Let g'_H be the truncation of the main geodesic g_H at μ_0 and η_0. Set $\mathbf{I}(g'_H) = \mu_0$ and $\mathbf{T}(g'_H) = \eta_0$. Given any $g'_Y \in H'$ with $D(g'_Y) = Y$, it follows from the definition of truncation that g'_Y is initially truncated from $g_Y \in H$ if and only if τ_{μ_0} contains some pair $(g'_Y, v_{Y,\mu_0}) \in \tau_{\mu_0}$. Since τ_{μ_0} is complete, repeated applications of property (S3) of slices gives a finite sequence of pairs $\{(g_{X_i}, x_i)\}_{i=1}^n$, with X_1 an annulus, $g_{X_n} = g_H$, $Y = X_k$ for some k, and $D(g_{X_i}) = X_i$ with X_i a component domain of (X_{i+1}, x_{i+1}) for each i. For each i, it follows from the definition of truncation that either g_{X_i} is initially truncated at x_i to a geodesic $g'_{X_i} \in H'$ with new initial vertex $v_{X_i,\mu_0} = x_i$, or x_i is the initial vertex of g_{X_i}. Either way, we may set $\mathbf{I}(g'_{X_{n-1}}) = \mu_0|_{X_{n-1}}$, and then inductively define $\mathbf{I}(g'_{X_i}) = \mathbf{I}(X_i, g'_{X_{i+1}}) = \mathbf{I}(g'_{X_{i+1}})|_{X_i}$; we note that each $\mathbf{I}(g'_{X_i})$ is a complete marking on X_i because μ_0 is a complete marking on S. Since each v_{X_i,μ_0} is the initial vertex of g'_{X_i}, it follows that $\mathbf{I}(X_i, g'_{X_{i+1}}) = \mathbf{I}(g'_{X_{i+1}})|_{X_i}$, which is complete and thus nonempty by induction. In particular, we have $\mathbf{I}(g'_Y) = \mathbf{I}(g_{X_i})$. It follows that $g'_H \swarrow g'_{X_{n-1}} \swarrow \cdots \swarrow g'_{X_1} \swarrow g'_Y$. We construct $\mathbf{T}(g'_Y)$ in a similar fashion in the case that g_Y is terminally truncated.

For the second case, suppose that $g'_Y \in H$ with $D(g'_Y) = Y$ and Γ_Y is not initially truncated. We need to perform an analysis similar to the proof of Lemma A.1, but truncation adds an extra wrinkle. Let $\tau_{Y,\mathrm{int}}$ be the slice in H which determines the initial marking of Γ_Y. Then $(g_Y, y) \in \tau_{Y,\mathrm{int}}$, where y is the initial vertex of g_Y. As before, repeated applications of

(S3) gives a sequence $\{(g_{X_i}, x_i)\}_{i=1}^n$ with $g_{X_n} = g_H$ and, for each i, $D(g_{X_i}) = X_i$ with X_i a component domain of (X_{i+1}, x_{i+1}). Since $\Gamma_Y \cap [\mu_0, \eta_0] \neq \emptyset$, it follows that there is at least $1 \leqslant m \leqslant n$ such that Γ_{X_m} is initially truncated, with each Γ_{X_k} initially truncated for $k \geqslant m$. Above, we defined $\mathbf{I}(g'_{X_{m-1}}) = \mathbf{I}(X_{m-1}, g'_{X_m}) = \mathbf{I}(g'_{X_m})|_{X_{m-1}}$, which is a complete marking on X_{m-1}. If x_{m-1} is not the initial vertex of $g'_{X_{m-1}}$, then we still have that $\mathbf{I}(X_{m-2}, g'_{X_{m-1}}) = \mathbf{I}(X_{m-2}, g_{X_{m-1}})$, which is nonempty by assumption, and we may define $\mathbf{I}(g'_{X_{m-2}}) = \mathbf{I}(g_{X_{m-2}}) = \mathbf{I}(X_{m-2}, g_{X_{m-1}}) = \mathbf{I}(X_{m-2}, g'_{X_{m-1}})$, implying that $g'_{X_{m-1}} \swarrow g'_{X_{m-2}}$. Otherwise, x_{m-1} is the initial vertex of $g'_{X_{m-1}}$, and we set $\mathbf{I}(g'_{X_{m-2}}) = \mathbf{I}(X_{m-2}, g'_{X_{m-1}}) = \mathbf{I}(g'_{X_{m-1}})|_{X_{m-2}}$, which is a complete marking on X_{m-2} and thus nonempty. Repeating this process, we can define $\mathbf{I}(g'_Y) = \mathbf{I}(Y, g'_{X_1})$ by induction. As in Lemma A.1, we find that $g'_{X_n} \swarrow \cdots \swarrow g'_Y$. We define $\mathbf{T}(g'_Y)$ similarly in the case where Γ_Y is not terminally truncated.

The truncated hierarchy. Let H_0 be the collection of the geodesics from H' with their marking data as constructed above. Note that every geodesic in H_0 is tight, as each is obtained by truncating a tight geodesic, truncation preserves the tightness property, and each geodesic has initial and terminal markings which respect the subordinancy relations. Thus H_0 is a collection of tight geodesics. Observe also that any subsurface $Y \subset S$ is the support of at most one geodesic in H_0, as this property holds for H by [**20**, Theorem 4.7(4)]. We now confirm that H_0 is a hierarchy by checking that it satisfies the three properties of Definition 4.2.

Lemma A.2. *The collection of tight geodesics H_0 is a hierarchy between μ_0 and η_0.*

Proof. We set $g'_H \in H_0$ to be the main geodesic of H_0, which has initial and terminal markings $\mathbf{I}(g'_H) = \mu_0$ and $\mathbf{T}(g'_H) = \eta_0$, respectively, thus satisfying property (H1). For property (H3), note that, for each $g'_Y \in H_0$, we have built $\mathbf{I}(g'_Y)$ and $\mathbf{T}(g'_Y)$ by first finding geodesics $g'_X, g'_Z \in H_0$ such that $g'_X \swarrow Y \searrow g'_Z$, and then defining $\mathbf{I}(g'_Y) = \mathbf{I}(Y, g'_X)$ and $\mathbf{T}(g'_Y) = \mathbf{T}(Y, g'_Z)$. In each case, we have shown these markings to be nonempty, implying that $g'_X \swarrow g'_Y \searrow g'_Z$. Thus (H3) is satisfied.

To see that property (H2) holds, let $g'_X, g'_Z \in H_0$ with $D(g'_X) = X, D(g'_Z) = Z$, and suppose $Y \subset S$ is a component domain of (X, x) and (Z, z) with $x \in g'_X$ and $z \in g'_Z$ such that $g'_X \swarrow Y \searrow g'_Z$. We need to prove that there exists a $g'_Y \in H_0$ with $g'_X \swarrow g'_Y \searrow g'_Z$. In the case where either Γ_X is initially truncated at x or Γ_Z is terminally truncated at z, we find g_Y in the slices for those points of truncation. We begin with the untruncated case, where we prove that g'_Y comes to us unscathed from H.

First, suppose that Γ_X and Γ_Z are not initially and terminally truncated at x and z, respectively, that is, x and z are not the initial and terminal vertices of g'_X and g'_Z, respectively. Then $\mathbf{I}(Y, g_X) = \mathbf{I}(Y, g'_X) \neq \emptyset$ and $\mathbf{T}(Y, g_Z) = \mathbf{T}(Y, g'_Z) \neq \emptyset$, which imply that $g_X \swarrow Y \searrow g_Z$. Since H is a hierarchy, it follows by definition that there is a unique geodesic $g_Y \in H$ with $D(g_Y) = Y$ and with $g_X \swarrow g_Y \searrow g_Z$.

Lemma A.1 implies that $(g_X, x) \in \tau_{Y,\mathrm{int}}$ and $(g_Z, z) \in \tau_{Y,\mathrm{ter}}$. We claim that $\tau_{\mu_0} <_s \tau_{Y,\mathrm{int}} \leqslant_s \tau_{Y,\mathrm{ter}} <_s \tau_{\eta_0}$, where $\tau_{Y,\mathrm{int}} =_s \tau_{Y,\mathrm{ter}}$ means $\tau_{Y,\mathrm{int}} = \tau_{Y,\mathrm{ter}}$. Assuming the claim, it follows from [**20**, Lemma 5.3] (see Lemma 5.6) that $\Gamma_Y \subset [\mu_0, \nu_0]$ and that g_Y is neither initially nor terminally truncated; thus $g_Y = g'_Y \in H_0$. Using the ending markings defined above, we have $\mathbf{I}(g'_Y) = \mathbf{I}(g_Y) = \mathbf{I}(Y, g_X) = \mathbf{I}(Y, g'_X)$ and $\mathbf{T}(g'_Y) = \mathbf{T}(g_Y) = \mathbf{T}(Y, g_Z) = \mathbf{T}(Y, g'_Z)$, implying $g'_X \swarrow g'_Y \searrow g'_Z$ by definition, completing the proof of this case.

We now prove the claim. Note that either $\tau_{Y,\mathrm{int}} = \tau_{Y,\mathrm{ter}}$ or [**20**, Lemma 5.3] implies $\tau_{Y,\mathrm{int}} <_s \tau_{Y,\mathrm{ter}}$. We prove that $\tau_{\mu_0} <_s \tau_{Y,\mathrm{int}}$; the proof that $\tau_{Y,\mathrm{ter}} <_s \tau_{\eta_0}$ is similar. By assumption, either g_X was initially truncated to g'_X at a vertex preceding x or it was not initially truncated; either way, there exists a vertex $x' \in g'_X$ with x' preceding x along g'_X, and thus g_X. By [**23**, Lemma 5.8], the pair (g_X, x') appears in some slice $\tau_{x'}$ along the resolution which gives Γ and, by our choice of x', we can choose $\tau_{x'}$ to determine a marking $\mu_{x'} \in [\mu_0, \eta_0]$. By definition of

$<_s$, we have $\tau_{x'} <_s \tau_{Y,\mathrm{int}}$ because $(g_X, x') <_p (g_X, x)$, and since $\tau_{\mu_0} \leqslant_s \tau_{x'}$ by [**20**, Lemma 5.3], we have $\tau_{\mu_0} <_s \tau_{Y,\mathrm{int}}$, as claimed.

Now suppose that Γ_X is initially truncated at x. Then $(g'_X, x) \in \tau_{\mu_0}$ and property (S3) of slices implies that there is a pair $(g_Y, y) \in \tau_{\mu_0}$ with $D(g_Y) = Y$. It follows then that $\mu_0 \in \Gamma_Y$; thus Γ_Y is initially truncated and there is $g'_Y \in H_0$ with $D(g'_Y) = Y$. Moreover, it follows from the inductive construction of $\mathbf{I}(g'_Y)$ above that $g'_X \swarrow g'_Y$. A similar argument implies $g'_Y \searrow g'_Z$ if Γ_Z is initially truncated at z. We note that g'_Y is unique because $g_Y \in H$ is unique by [**20**, Theorem 4.7(4)].

There are two mixed cases, where either Γ_X or Γ_Z is truncated, but not both; each can be handled in the same fashion as the other. In the case where Γ_X is truncated at x, we have already shown that there are $g_Y \in H$ and $g'_Y \in H_0$ with $D(g_Y) = D(g'_Y) = Y$ such that $g'_X \swarrow g'_Y$. We have also shown that $Y \searrow g_Z$ since Γ_Z is not truncated at z. Since Y supports a geodesic $g_Y \in H$, [**20**, Theorem 4.7(1)] implies that $g_Y \searrow g_Z$, and it follows from the above argument that $g'_Y \searrow g'_Z$. □

Resolving the truncated hierarchy. Having proved that H_0 is a hierarchy, we can now prove the following lemma.

LEMMA A.3. *The resolution of slices $\tau_\mu = \tau_1 \to \cdots \to \tau_k = \tau_\eta$ of H is also a resolution of slices of H_0.*

Proof. First of all, it follows from the definitions that each slice in the above resolution is a complete slice on H_0. It suffices to prove that each move $\tau_i \to \tau_{i+1}$ is an elementary move of slices.

Since $\tau_i \to \tau_{i+1}$ is an elementary move along some geodesic $g_V \in H$ from v to v' where $v, v' \in g_V$, there are initial and terminal transition slices, σ and σ', respectively, such that $\sigma \subset \tau_i$, $\sigma' \subset \tau_{i+1}$, and $\tau_i \setminus \sigma = \tau_{i+1} \setminus \sigma'$. Any geodesic g_X involved in τ_i or τ_{i+1} has a truncation $g'_X \in H_0$ by definition. Let $Y \subset S$ be such that $Y|_{v'} \neq \emptyset$ so that $(g_Y, y) \in \sigma$, where y is the terminal vertex of g_Y. Then it follows from the definition of σ that τ_i is the terminal slice of Γ_Y. As such, Γ_Y is not terminally truncated at y and y is the terminal vertex of g'_Y, putting (g'_Y, y) in the H_0 initial transition slice from τ_i to τ_{i+1}. Similarly, if $Z|_v \neq \emptyset$ so that $(g_Z, z) \in \sigma'$, then τ_{i+1} is the initial slice of Γ_Z and (g'_Z, z) is in the H_0 terminal transition slice from τ_i to τ_{i+1}; that is, σ and σ' are the H_0-transition slices for $\tau_i \to \tau_{i+1}$, proving that it is an elementary move in H_0.

This proves that $\tau_\mu = \tau_1 \to \cdots \to \tau_k = \tau_\eta$ is a resolution of slices of H_0. □

Thus we have shown the following proposition.

PROPOSITION A.4. *The subpath $[\mu_0, \eta_0] \subset \Gamma$ is a hierarchy path based on H_0. In particular, subpaths of hierarchy paths are hierarchy paths.*

As an immediate corollary of Proposition A.4 and [**20**, Theorem 6.10], we have the following corollary.

COROLLARY A.5. *Hierarchy paths are uniform quasigeodesics in $\mathcal{M}(S)$.*

REMARK A.6. The fact that hierarchy paths are uniform quasigeodesics is well known to the experts, but has not, to our knowledge, ever been recorded. We note that Proposition A.4 is a stronger statement than necessary for this fact.

A.4. *Structure of active segments*

Given a hierarchy path Γ based on a hierarchy H between $\mu, \eta \in \mathcal{M}(S)$ and a nonannular subsurface Y with nonempty active segment Γ_Y, every marking $\mu \in \Gamma_Y$ naturally restricts to a complete, clean marking $\mu|_Y \in \mathcal{M}(Y)$. In the case that Y is an annulus with core α, $\mu|_Y = t_\alpha$, where t_α is the transversal to α in μ. In this subsection, we prove that the restriction of Γ_Y to $\mathcal{M}(Y)$ coincides with a hierarchy path naturally defined from the restricted hierarchy for Γ_Y constructed in Proposition A.4. For the purposes of this subsection, a hierarchy and hierarchy path on an annular domain are just a geodesic.

By Proposition A.4, we may consider Γ_Y as a hierarchy path based on H', so we may suppose without loss of generality that $\Gamma = \Gamma_Y$, $H = H'$, and $\mu_{Y,\mathrm{int}} = \mu$ and $\mu_{Y,\mathrm{ter}} = \eta$. Let $H_Y = \{g_Z \in H \mid Z \subseteq Y\}$ be the collection of all tight geodesic in H supported on subsurfaces of Y with the same initial and terminal markings as in H. Note that if $g_Z \in H_Y$ with $D(g_Z) = Z \subset Y$, then $\mathbf{I}(g_Z)|_Z = \mathbf{I}(g_Z)$ and $\mathbf{T}(g_Z)|_Z = \mathbf{T}(g_Z)$.

LEMMA A.7. *H_Y is a hierarchy between $\mu_Y = \mu|_Y$ and $\eta_Y = \eta|_Y$.*

Proof. In the case that Y is an annulus with core α, $H_Y = \{g_Y\}$, and the conclusion is obvious. Suppose that Y is nonannular. Let $g_Y \in H_Y$ be the base geodesic of H_Y, with $\mathbf{I}(g_Y) = \mu_Y$ and $\mathbf{I}(g_Y) = \eta_Y$ by definition. Let $\tau_{\mathrm{int}} \to \cdots \to \tau_{\mathrm{ter}}$ be the sequence of elementary moves of slices which give Γ. Let $g_Z \in H'$ with $Z \subset Y$ and suppose $(g'_Z, z) \in \tau_{Z,\mathrm{int}}$, where $\tau_{Z,\mathrm{int}}$ is a initial slice of the active segment of Z along Γ, namely Γ_Z . Since $\Gamma = \Gamma_Y$, there is a $y \in g_Y$ with $(g_Y, y) \in \tau$, and Lemma A.1 implies that there is a sequence of $\{g_{X_i}\}_{i=1}^n \subset H$, with $X_n = Y$, $X_1 = Z$, and $g_Y \swarrow g_{X_{n-1}} \swarrow \cdots \swarrow g_Z$. Similarly, $g_Z \searrow \cdots \searrow g_Y$. In particular, all geodesics in H_Y other than g_Y are directly forward and backward subordinate to other geodesics in H_Y. It follows easily from the definitions that H_Y is a hierarchy between μ_Y and η_Y. □

Consider the resolution $\tau_\mu = \tau_1 \to \cdots \to \tau_N = \tau_\eta$ of slices of H which gives Γ. For each τ_i in this resolution, let $\mu_i \in \Gamma$ be its corresponding marking and set $\tau_{Y,i} = \{(g_Z, z) \mid (g_Z, z) \in \tau_i \text{ and } g_Z \in H_Y\}$. The set of $\{\tau_{Y,i}\}_{i=1}^N$ possibly contains redundancies corresponding to elementary moves along $\tau_\mu = \tau_1 \to \cdots \to \tau_N = \tau_\eta$, which make progress on geodesics whose domains of support are not contained in Y; removing these redundancies and relabeling as necessary gives a sequence of slices $\tau_{\mu_Y} = \tau_{Y,1} \to \cdots \to \tau_{Y,N'} = \tau_{\eta_Y}$. We may similarly reparameterize $\mu|_Y = (\mu_1)|_Y \to \cdots \to (\mu_N)|_Y = \eta_Y$ to $\mu_Y = \mu_{Y,1} \to \cdots \to \mu_{Y,N'} = \eta_Y$, which we denote by $(\Gamma_Y)|_Y$. It follows from the definitions that $(\mu_i)|_Y$ is compatible with $\tau_{Y,i}$.

LEMMA A.8. *The sequence $\mu_Y = \mu_{Y,1} \to \cdots \to \mu_{Y,N'} = \eta_Y$ is a hierarchy path based on H_Y.*

Proof. If Y is an annulus, then $\mu_Y = \mu_{Y,1} \to \cdots \to \mu_{Y,N'} = \eta_Y$ is the geodesic g_Y, satisfying the claim. Otherwise, it suffices to show that $\tau_{Y,i} \to \tau_{Y,i+1}$ is an elementary move on slices of H_Y for each $1 \leqslant i \leqslant N' - 1$. Each such pair $\tau_{Y,i} \to \tau_{Y,i+1}$ is restricted from an elementary move of slices $\tau_j \to \tau_{j+1}$. Since τ_j and τ_{j+1} are complete slices on S, it follows that $\tau_{Y,i}$ and $\tau_{Y,i+1}$ are complete slices on Y. Having removed redundancies, $\tau_j \to \tau_{j+1}$ realizes forward progress from z to z' along some geodesic $g_Z \in H_Y$. Let $\sigma \subset \tau_j$ and $\sigma' \subset \tau_{j+1}$ with $\tau_j \setminus \sigma = \tau_{j+1} \setminus \sigma'$ be the transition slices for $\tau_j \to \tau_{j+1}$. By definition [**20**, Section 5], the domains supporting geodesics σ and σ' are component domains of $g_Z \setminus z'$ and $g_Z \setminus z$, respectively. It follows from the definition that $\sigma \subset \tau_{Y,i}$ and $\sigma' \subset \tau_{Y,i+1}$ with $\tau_{Y,i} \setminus \sigma = \tau_{Y,i+1} \setminus \sigma'$ are the transition slices of the transition from z to z' along g_Z in H_Y. Thus $\tau_{Y,i} \to \tau_{Y,i+1}$ is an elementary move of slices in H_Y, completing the proof. □

Combined with Proposition A.4, we have the following proposition about the structure of active segments of hierarchy paths, which resembles [**26**, Theorem 5.3] for Teichmüller geodesics.

PROPOSITION A.9 (The structure of active segments). *Let $K > 0$ be the large link constant from Lemma 2.5 and Γ be a hierarchy path based on a hierarchy H. Let $\Gamma_Y \subset \Gamma$ be the active segment of $g_Y \in H$ with $D(g_Y) = Y \subset S$ and H_Y be the corresponding restricted hierarchy in $\mathcal{M}(Y)$. Then the following hold.*

(i) *For any segment $[\mu_0, \eta_0] \subset \Gamma$ with $[\mu_0, \eta_0] \cap \Gamma_Y = \emptyset$, we have $d_Y(\mu_0, \eta_0) < K$.*

(ii) *The restriction of Γ_Y to $\mathcal{M}(Y)$ can be reparameterized to a hierarchy path based on H_Y.*

Proof. Let $\Gamma_1 = [\mu, \mu_1], \Gamma_2 = [\mu_2, \eta] \subset \Gamma$ be the two components of $\Gamma \setminus \Gamma_Y$. These are both hierarchy paths by Proposition A.4, based on hierarchies H_1 and H_2, respectively. Since g_Y is in both H_Y and H, it follows that neither H_1 nor H_2 contains a geodesic supported on Y. Thus Lemma 2.5 implies that $d_Y(\mu, \mu_1), d_Y(\mu_2, \eta) < K$, completing the proof of (1).

(2) follows directly from Lemmata A.7 and A.8. □

Acknowledgements. The author would like to thank Daniel Groves for his great encouragement and guidance. He would also like to thank to Jonah Gaster, Hao Liang, Howard Masur, Yair Minsky, and Kasra Rafi for interesting conversations. Finally, the author would like to thank the anonymous referee for several useful comments.

References

1. J. BEHRSTOCK, 'Asymptotic geometry of the mapping class group and Teichmuller space', *Geom. Topol.* 10 (2006) 1523–1578.
2. J. BEHRSTOCK, C. DRUTU and L. MOSHER, 'Thick metric spaces, relative hyperbolicity, and quasi-isometric regidity', *Math. Ann.* 344 (2009) 543–595.
3. J. BEHRSTOCK, M. HAGEN and A. SISTO, 'Hierarchically hyperbolic spaces I: curve complexes for cubical groups', Preprint, 2014, arXiv:1312:2171.
4. J. BEHRSTOCK, B. KLEINER, Y. MINSKY and L. MOSHER, 'Geometry and rigidity of mapping class groups', *Geom. Topol.* 16 (2012) 781–888.
5. J. BEHRSTOCK and Y. MINSKY, 'Dimension and rank for mapping class groups', *Ann. Math.* 167 (2008) 1055–1077.
6. B. BOWDITCH, 'The asymptotic geometry of the Teichmüller metric', *J Topol.* Preprint, 2014.
7. J. BROCK, 'The Weil–Petersson metric and volumes of 3-dimensional hyperbolic convex cores', *J. Amer. Math. Soc.* 16 (2003) 495–535 (electronic).
8. J. BROCK, R. CANARY and Y. MINSKY, 'The classification of Kleinian surface groups II: the ending lamination conjecture', *Ann. Math.* 176 (2012) 1–149.
9. J. BROCK and B. FARB, 'Curvature and rank of Teichmüller space', *Amer. J. Math.* 128 (2006) 1–22.
10. J. W. CANNON and D. COOPER, 'A characterization of cocompact hyperbolic and finite-volume hyperbolic groups in dimension three', *Trans. Amer. Math. Soc.* 330 (1992) 419–431.
11. M. DURHAM, 'The asymptotic geometry of the Teichmüller metric: dimension and rank', Preprint, 2015, arXiv:1501.0020.
12. A. ESKIN, H. MASUR and K. RAFI, 'Large scale rank of Teichmüller space', Preprint, 2013, arXiv:1307.3733.
13. B. GROFF, 'Quasi-isometries, boundaries, and JSJ-decompositions of relatively hyperbolic groups', Preprint, 2012, arXiv:1210.1166v3.
14. D. GROVES and J. F. MANNING, 'Dehn filling in relatively hyperbolic groups', *Israel J. Math.* 168 (2008) 317–429.
15. W. J. HARVEY, 'Boundary structure of the modular group', *Riemann surfaces and related topics: proceedings of the 1978 stony brook conference* (*State University New York, Stony Brook, NY,* 1978), Annals of Mathematics Studies 97 (Princeton University Press, Princeton, NJ, 1981) 245–251.
16. S. KERCKHOFF, 'The asymptotic geometry of Teichmüller space', *Topology* 19 (1980) 23–41.
17. J. MANGAHAS, 'Uniform uniform exponential growth of the mapping class group', *Geom. Funct. Anal.* 19 (2010) 1468–1480.
18. B. MASKIT, 'Comparison of hyperbolic and extremal lengths', *Ann. Acad. Sci. Fenn. Ser. A I Math.* 10 (1985) 381–386.

19. H. Masur and Y. N. Minsky, 'Geometry of the complex of curves. I. Hyperbolicity', *Invent. Math.* 138 (1999) 103–149.
20. H. Masur and Y. N. Minsky, 'Geometry of the complex of curves. II. Hierarchical structure', *Geom. Funct. Anal.* 10 (2000) 902–974.
21. Y. N. Minsky, 'Teichmüller geodesics and ends of hyperbolic 3-manifolds', *Topology* 32 (1993) 625–647.
22. Y. N. Minsky, 'Extremal length estimates and product regions in Teichmüller space', *Duke Math. J.* 83 (1996) 249–286.
23. Y. N. Minsky, 'The classification of Kleinian surface groups, I: Models and bounds', *Ann. of Math.* (2) 171 (2010) 1–107.
24. K. Rafi, 'A characterization of short curves of a Teichmüller geodesic', *Geom. Topol.* 9 (2005) 179–202.
25. K. Rafi, 'A combinatorial model for the Teichmüller metric', *Geom. Funct. Anal.* 17 (2007) 936–959.
26. K. Rafi, 'Hyperbolicity in Teichmüller Space', *Geom. Topol.* 18 (2014) 3025–3053.
27. J. Tao, 'Linearly bounded conjugator property for mapping class groups', *Geom. Funct. Anal.* 23 (2013) 415–466.
28. T. Aougab and S. Taylor, 'Richard Webb – Effective Masur-Minsky distance formulas and applications to hyperbolic 3-manifolds', Preprint, 2016, Yale University.

Matthew Gentry Durham
Department of Mathematics
University of Michigan
2074 East Hall
530 Church Street
Ann Arbor, MI 48109-1043
USA

durhamma@umich.edu